1.4 Maxima and Minima

Suppose that $f(x,y)$ and its first and second partial derivatives are continuous at (a,b) (critical point) and $f_x(a,b) = f_y(a,b) = 0$, then

(i) $f(x,y)$ has a local maximum at (a,b) if

$$f_{xx}(a,b) < 0 \text{ and } f_{xx}(a,b)f_{yy}(a,b) - \Big[f_{xy}(a,b)\Big]^2 > 0.$$

(ii) $f(x,y)$ has a local minimum at (a,b) if

$$f_{xx}(a,b) > 0 \text{ and } f_{xx}(a,b)f_{yy}(a,b) - \Big[f_{xy}(a,b)\Big]^2 > 0.$$

(iii) $f(x,y)$ has a saddle point at (a,b) if

$$f_{xx}(a,b)f_{yy}(a,b) - \Big[f_{xy}(a,b)\Big]^2 < 0.$$

Note that to find the critical points of $f(x,y)$, we suppose both $f_x(x,y)$ and $f_y(x,y) = 0$, then we solve the equation to x and y.
Example: Find local maxima, local minima and saddle points of the functions

(i) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

(ii) $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

(iii) $f(x,y) = x^2 + xy + y^2 + 2y + 5$

Solution
(i) We need firstly to find the critical points of $f(x,y)$, where

$$f_x(x,y) = 2x + y + 3 = 0 \text{ and } f_y(x,y) = x + 2y - 3 = 0.$$

Thus, there is one critical point $(-3,3)$. Also, we have

$$f_{xx}(-3,3) = 2, f_{yy}(-3,3) = 2, \text{ and } f_{xy}(-3,3) = 1$$

so $f_{xx}(-3,3) = 2 > 0$ and

$$f_{xx}(-3,3)f_{yy}(-3,3) - \Big[f_{xy}(-3,3)\Big]^2 > 0$$
$$(2)(2) - (1)^2 = 3 > 0.$$

Thus, $f(x, y)$ has a local minimum at $(-3, 3)$ which is $f(-3, 3) = -5$.

(ii) We need firstly to find the critical points of $f(x, y)$, where

$$f_x(x, y) = y - 2x - 2 = 0 \text{ and } f_y(x, y) = x - 2y - 2 = 0.$$

Thus, there is one critical point $(-2, -2)$. Also, we have

$$f_{xx}(-2, -2) = -2, f_{yy}(-2, -2) = -2, \text{ and } f_{xy}(-2, -2) = 1$$

so $f_{xx}(-2, -2) = -2 < 0$ and

$$f_{xx}(-2, -2)f_{yy}(-2, -2) - \Big[f_{xy}(-2, -2)\Big]^2 > 0$$
$$(-2)(-2) - (1)^2 = 3 > 0.$$

Thus, $f(x, y)$ has a local maximum at $(-2, -2)$ which is $f(-2, -2) = 8$.

(iii) We need firstly to find the critical points of $f(x, y)$, where

$$f_x(x, y) = 2x + y + 3 = 0 \text{ and } f_y(x, y) = x + 2 = 0.$$

Thus, there is one critical point $(-2, 1)$. Also, we have

$$f_{xx}(-2, 1) = 2, f_{yy}(-2, 1) = 0, \text{ and } f_{xy}(-2, 1) = 1$$

so

$$f_{xx}(-2, 1)f_{yy}(-2, 1) - \Big[f_{xy}(-2, 1)\Big]^2 < 0$$
$$(2)(0) - (1)^2 = -1 < 0.$$

Thus, $f(x, y)$ has a saddle point at $(-2, 1)$.

1.5 Exercises

Q_1 : **Find the limits of the functions below**

$$\lim_{(x,y)\longrightarrow(0,4)} \frac{x}{\sqrt{y}}$$

$$\lim_{(x,y)\longrightarrow(1,1)} \frac{x^2-y^2}{x-y}$$

$$\lim_{(x,y)\longrightarrow(2,2)} \frac{x+y-4}{\sqrt{x+y}-2}$$

$$\lim_{(x,y)\longrightarrow(3,4)} \frac{\sqrt{x}-\sqrt{y-1}}{x-y-1}$$

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{\sqrt{x^2-2y^3-2xy^2+xy}}{\sqrt{x+y}}$$

Q_2 : **Find** $\frac{\partial z}{\partial x}$ **and** $\frac{\partial z}{\partial y}$

(i) $z = 5xy - 7x^2 - y^2 + 3x - 6y$

(ii) $z = \frac{x}{x^2+y^2}$

(iii) $z = e^{xy}\ln(y)$

(v) $z = tan^{-1}(\frac{y}{x})$

Q_3 : **Verify that** $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$ **if**

(i) $F = x\sin(y) + y\sin(x) + xy$

(ii) $F = \ln(2x+3y)$

(iii) $F = e^x + x\ln(y) + y\ln(x)$

Q_4 : **Prove that**

(i) If $G(r,\Theta) = \sqrt{r^2+\Theta^2}$, then $rG_r + \Theta G_\Theta = G$.

(ii) If $W = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then $xW_x + yW_y + zW_z = 0$.

(iii) If $V = f(s+t) + sg(s+t)$, then $V_{ss} - V_{st} + V_{tt} = 0$.

Q_5 : **Find local maxima, local minima and saddle points of the functions below**

(1) $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

(2) $f(x,y) = 3y^2 - 3x^2 - 2y^3 - 3x^2 + 6xy$

(3) $f(x,y) = x^3 + 3x^2 + y^3 - 3y^2 - 8$