Chapter 2

Double Integrals

In this chapter, we will learn how to evaluate double integrals. Let $z = f(x, y)$ be a function which is continuous on closed region $D : a_1(y) \leq x \leq a_2(y), b_1(x) \leq y \leq b_2(x)$. We may interpret the double integral of z over D as the volume. So, we define this volume to be

$$\text{volume} = \int\int_D f(x,y)dA = \int_{y=b_1(x)}^{y=b_2(x)} \int_{x=a_1(y)}^{x=a_2(y)} f(x,y)dxdy.$$

Or

$$\text{volume} = \int\int_D f(x,y)dA = \int_{x=a_1(y)}^{x=a_2(y)} \int_{y=b_1(x)}^{y=b_2(x)} f(x,y)dydx.$$

2.1 Properties of Double integrals

If $f(x, y)$ and $g(x, y)$ are continuous on the region D then

$$\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA \text{ for any number } c.$$

$$\int\int_D \Big[f(x,y) \mp g(x,y)\Big]dA = \int\int_D f(x,y)dA \mp \int\int_D g(x,y)dA.$$

$$\int\int_D f(x,y)dA = \int\int_{D_1} f(x,y)dA + \int\int_{D_2} f(x,y)dA,$$

if D is the union of two regions $D = D_1 \cup D_2$.
Note that if $f(x, y)$ is continuous over the region $D : a \leq x \leq b, c \leq y \leq d$, then

$$\int\int_D f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

Example: Calculate

$$\int\int_D f(x,y)dA \text{ for } f(x,y) = 1 - 6x^2y \text{ and } D : 0 \le x \le 2, -1 \le y \le 1.$$

Solution

$$\begin{aligned}
\int\int_D f(x,y)dA &= \int_{-1}^{1}\int_{0}^{2}(1-6x^2y)dxdy \\
&= \int_{-1}^{1}\left[x - 2x^3y\right]_0^2 dy, \\
&= \int_{-1}^{1}(2-16y)dy, \\
&= \left[2y - 8y^2\right]_{-1}^{1} \\
&= 4.
\end{aligned}$$

Now by changing the order of integration gives the same answer:

$$\begin{aligned}
\int_0^2\int_{-1}^{2}(1-6x^2y)dydx &= \int_0^2\left[y - 3x^2y^2\right]_{-1}^{1}dx, \\
&= \int_0^2[(1-3x^2)-(-1-3x^2)]dx \\
&= 4.
\end{aligned}$$

Example: Find the volume of solid bounded by the lines $x = 4$ and $y = 8$ and the surface $z = 4 - \frac{x}{2} + \frac{y^2}{16}$.

Solution

By drawing (3-dimensional) the lines $x = 4$ and $y = 8$ plane and the surface $z = 4 - \frac{x}{2} + \frac{y^2}{16}$ which gives (see Figure 1)

$$
\begin{aligned}
\text{volume} &= \int\int_D z dA \\
&= \int_0^4 \int_0^8 (4 - \frac{x}{2} + \frac{y^2}{16}) dy dx \\
&= \int_0^4 \left[4y - \frac{yx}{2} + \frac{y^3}{48} \right]_0^8 dx, \\
&= \int_0^2 (\frac{128}{3} - 2x^2) dx \\
&= 416/3.
\end{aligned}
$$

Figure 1

Example: Find the volume of solid bounded by the lines $x = 2$ and $y = 1$ and the plane $z = 4 - x - y$.
Solution
By drawing the lines $x = 1$ and $y = 1$ and the plane
$z = 4 - x - y$ which gives (see Figure 2)

$$
\begin{aligned}
\text{volume} &= \int\int_D z dA \\
&= \int_0^2 \int_0^1 (4 - x - y) dy dx \\
&= \int_0^1 \left[4y - xy + \frac{y^2}{2} \right]_0^2 dx, \\
&= \int_0^2 (\frac{7}{2} - x) dx \\
&= 5.
\end{aligned}
$$

Figure 2

2.2 Areas and Centers of Mass

In this section, we show how to use double integrals to calculate the areas of bounded regions in the plane. Also, we study the physical problem of finding the center of mass of a thin flat plate covering a region in plane.