Polar integration formula

$$\int\int_D f(x,y)dA = \int\int_G (r\cos\theta, r\sin\theta)\, r dr d\theta \text{ where } G : \alpha \leq \theta \leq \beta, b(\theta) \leq r \leq a(\theta).$$

Note that for all point (r,θ) in the region G:

$$r \geq 0 \text{ and } 0 \leq \theta \leq 2\Pi.$$

Example: Evaluate $\int\int_D e^{x^2+y^2} dydx$, where D is region bounded by the $x-$ axis and the curve $y = \sqrt{1-x^2}$?

Solution

There is no way to integrate $e^{x^2+y^2}$ with respect to either x or y. Thus, we need to use the polar coordinates which enables us to evaluate the integral as

$$y = \sqrt{1-x^2} \Longrightarrow y^2 = 1-x^2 \Longrightarrow x^2+y^2 = 1 \Longrightarrow r^2 = 1 \Longrightarrow r = \mp 1 \Longrightarrow r = 1$$

We sketch the region $y = \sqrt{1-x^2}$ and $x-$axis (see Figure 7) so

$$\begin{aligned}
\int\int_D e^{x^2+y^2} dydx &= \int_0^{\Pi}\int_0^1 (re^{r^2}) dr d\theta \\
&= \int_0^{\Pi}\left[\frac{1}{2}e^r\right]_0^1 d\theta, \\
&= \frac{1}{2}\int_0^1 (e-1) d\theta \\
&= \frac{\Pi}{2}(e-1).
\end{aligned}$$

Figure 7

Example: Evaluate $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}} dydx.$

Solution

There is no way to integrate $\frac{2}{1+\sqrt{x^2+y^2}}$ with respect to either x or y. By using a polar coordinates gives

$$y = -\sqrt{1-x^2} \Longrightarrow y^2 = 1-x^2 \Longrightarrow x^2+y^2 = 1 \Longrightarrow r^2 = 1 \Longrightarrow r = \mp 1 \Longrightarrow r = 1$$

We sketch $y = -\sqrt{1-x^2}, y = 0, x = 0$ and $x = -1$ (see Figure 8) so

$$\begin{aligned}
\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}} &= \int_{\Pi}^{3\frac{\Pi}{2}}\int_{0}^{1} \frac{2r}{1+r} dr d\theta \\
&= \int_{\Pi}^{3\frac{\Pi}{2}} \Big[r - \ln(r+1)\Big]_{0}^{1} d\theta, \\
&= \frac{1}{2}\int_{0}^{1}(1-\ln 2)d\theta \\
&= (1-\ln 2)\Pi.
\end{aligned}$$

Figure 8

Example: Evaluate by using polar integral $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)dydx.$

Solution

2.4 Exercises

Q_1 : **Evaluate each of the following double integral**

(1) $\int\int_D f(x,y)dA$ for $f(x,y) = x^2y - 2xy$ and $D : 0 \leq x \leq 3, -2 \leq y \leq 0$.

(2) $\int\int_D f(x,y)dA$ for $f(x,y) = x \sin y$ and $D : 0 \leq x \leq \Pi, 0 \leq y \leq x$.

(3) $\int\int_D f(x,y)dA$ for $f(x,y) = e^{x+y}$ and $D : 0 \leq x \leq \ln y, 1 \leq y \leq \ln 8$.

Q_2 : **Find the volume of the solid cut from the first actant by the surface** $z = 4 - x^2 - y$**?**

Q_3 : **Find the area of the region** D **for each of following by using the double integral**

(1) D : The lines $x + y = 2, x = 0$ and $y = 0$.

(2) D : The parabola $x = -y^2$ and the line $y = x + 2$.

(3) D : The curve $y = e^x$ and the lines $y = 0, x = 0$ and $x = \ln 2$.

Q_4 : **Evaluate**

(1) $\int_0^1 \int_y^1 \frac{\sin x}{x} dxdy$

(2) $\int_0^1 \int_x^1 e^{y^2} dydx$

(3) $\int_0^3 \int_{x^2}^9 x \cos y^2 dydx$

(4) $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} dxdy$

Q_5 : **Find a center of mass of a thin plat of density** $\delta(x,y) = 3$ **bounded by** $y = 2 - x^2, y = x$ **and** $y-$ **axis?**

Q_6 : **Evaluate by using polar integral** $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$.