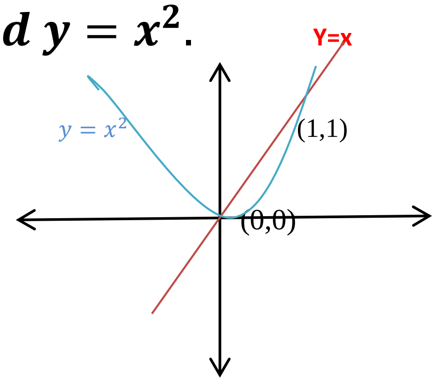
Using Green's Theorem to Evaluate line integral

If $\boldsymbol{M(x,y)}$ **and** $\boldsymbol{N(x,y)}$ **having continuous first partial derivatives in an open region containing** $\boldsymbol{R}$**. Then**

$$\oint_C \ M(x,y)dx + \ N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Where C is a positively simple closed curves

<u>Example</u>: **Verify Green's Theorem for** $\oint_C \ y^2dx + x^2dy$**, where** C **is the closed curve of the region bounded by** $y = x \ and \ y = x^2$**.**



Solution :

$$I = I_1 + I_2$$

Path one (I_1): when $y = x^2 \ \rightarrow dy = 2xdx$ so,

$$\oint_C \ y^2dx + x^2dy \Longrightarrow \int_0^1 x^4dx + 2x^2dx = \frac{x^5}{5} + \frac{x^4}{2} \ |_0^1 = \frac{7}{10}$$

Path two (I_2): when $y = x \ \rightarrow dy = xdx$ so,

$$\oint_C \ y^2dx + x^2dy \Longrightarrow \int_1^0 x^2dx + x^2dx = \frac{2x^3}{3} \ |_1^0 = \frac{-2}{3}$$

Thus, $\oint_C \ y^2dx + x^2dy = \frac{7}{10} - \frac{2}{3} = \frac{1}{30}$

Now, we can use **Green's Theorem** to change the line integral into a double integral over the region R:

Taking: $\boldsymbol{M(x,y) = y^2 and \ N(x,y) = x^2}$

$$\frac{\partial M(x,y)}{\partial y} = 2y \quad and \ \frac{\partial N(x,y)}{\partial x} = 2x$$

$$\int_0^1 \int_{x^2}^{x} (2x - 2y)dydx = \int_0^1 (2xy - y^2) \ |_{x^2}^{x} dx = \int_0^1 x^2 - 2x^3 - x^4 \, dx = \frac{1}{30}$$

$$\oint_C \ y^2dx + x^2dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \frac{1}{30}$$

Example: **Verify Green's Theorem for $\oint_C x^2 dx + xy dy$, where C is the triangle having vertices $(0,0), (1,0), and\ (1,1)$.**

Solution: H.W

Exercises

Q1) Verify Green's Theorem for $\oint_C x^2 dx + xy dy$, where C is the triangle having vertices $(0,2), (2,0), and\ (4,2)$.

Q2) Verify Green's Theorem for $\oint_C 2xy^3 dx + 4x^2y^2 dy$, Where C is the closed curve of the region bounded by: $x - axis$, $x = 1\ and\ y = x^3$.

Q3) Verify Green's Theorem for $\oint_C \cos y\, dx + e^x dy$

Where C is the triangle having vertices $(0,0), (\pi,0), and\ (\pi,\pi)$.

Q4) Verify Green's Theorem for $\oint_C xy dx + 2dy$ Where C is the closed curve of the region bounded by: $x - axis\ and\ y = \sqrt{4 - x^2}$.

Q5) Apply Green's Theorem for $\oint_C xy dy - y^2 dx$ where C is the square cut from the first quadrant by $x = 1\ and\ y = 1$.