Using Chain Rule with Partial Derivatives

The Chain Rule for functions of a single variable studied in first stage says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and $\frac{dw}{dt}$ can be calculated by the formula $\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$.

For functions of two or more variables the Chain Rule has several forms. The form depends on how many variables are involved, but there are three general cases as a following:

Case1: If $w = f(x, y, z)$ is a differentiable function and $x = x(t), y = y(t), and\ z = z(\text{t}),$ are differentiable functions of t, then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example: Prove that $\frac{dw}{dt} = \cos 2t$, if $w = xy, and\ x = \cos t\,, y = \sin t$.

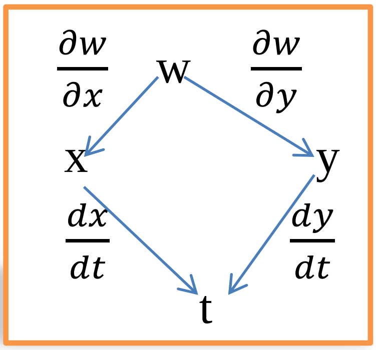
Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

Note that $\frac{\partial w}{\partial x} = y\,, \frac{dx}{dt} = -\sin t\,, \frac{\partial w}{\partial y} = x, and\ \frac{dy}{dt} = \cos t$.

Now, $\frac{dw}{dt} = -y \sin t + x \cos t,$

$$\frac{dw}{dt} = -\sin t \sin t + \cos t \cos t,$$

$$\frac{dw}{dt} = cos^2 t - sin^2 t = \cos 2t.$$



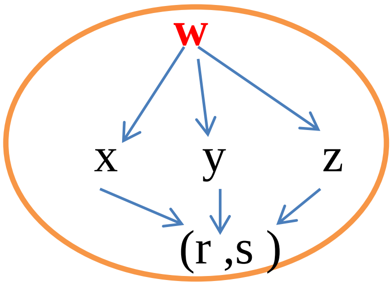
Case2: If $w = f(x, y, z)$ where $x = g(r, s), y = h(r, s), and\ z = k(r, s)$. If all four functions are differentiable functions, then w have partial derivatives with respect to r and s given by the formulas:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Example: Find $\frac{\partial w}{\partial r}$ *and* $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2 + z^2$, *and* $x = r^2 + s$, $y = r\ln s$, $z = \tan^{-1}(s^2 + r^2)$.

Solution: By using the formulas in case2



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial r} = (2x)\,(2r) + (2y)\,(\ln s) + (2z)\left(\frac{2r}{1+(s^2+r^2)^2}\right),$$

$$\frac{\partial w}{\partial r} = (2r^2 + 2s)\,(2r) + (2r\ln s)\ln s + (2\tan^{-1}(s^2 + r^2))\left(\frac{2r}{1+(s^2+r^2)^2}\right),$$

$$\frac{\partial w}{\partial r} = (4r^3 + 4rs) + (2r\ln^2 s) + \left(\frac{4r\tan^{-1}(s^2+r^2)}{1+(s^2+r^2)^2}\right)$$

Now,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s},$$

$$\frac{\partial w}{\partial s} = (2x)\,(1) + (2y)\left(\frac{r}{s}\right) + (2z)\left(\frac{2s}{1+(s^2+r^2)^2}\right)$$

$$\frac{\partial w}{\partial s} = 2\,(r^2 + s) + (2r\ln s)\frac{r}{s} + (2\tan^{-1}(s^2 + r^2))\left(\frac{2s}{1+(s^2+r^2)^2}\right)$$

$$\frac{\partial w}{\partial s} = (2r^2 + 2s) + \frac{2r^2\ln s}{s} + \left(\frac{4s\tan^{-1}(s^2+r^2)}{1+(s^2+r^2)^2}\right)$$

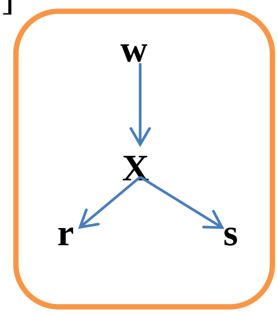
Case3: If $w = f(x)$ and $x = g(r,s)$, then w has partial derivatives with respect to r and s given by the formulas:

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

Example: If $w = f(x)$, *and* $x = r - s$, then show that $\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial s} = -[f'(x)]^2$

Solution: By using the formulas in case3



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}(1) = [f'(x)]$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}(-1) = -f'(x)$$

Thus, $\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial s} = -[f'(x)]^2$

1) Find $\frac{dw}{dt}$ if $\ln(z^2 + x^2 + y^2)$, $x = t^2$, $y = e^{t^2}$, $z = \sin^{-1} t$

2) Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$ if $f(x, y, z) = \sqrt{z^2 + x^2 + y^2}$,

$x = e^r \cos(s)$, $y = e^r \sin(s)$, $z = e^r$

3) If $w = f(s)$ and $s = y + ax$, then show that $\frac{\partial w}{\partial x}$ -a$\frac{\partial w}{\partial y}$=0

4) If $w = f(x, y)$ $x = r\cos(s)$, $y = r\sin(s)$, then show that

$\left(\frac{\partial w}{\partial r}\right)^2$ -r^{-2} $\left(\frac{\partial w}{\partial s}\right)^2$=$\left(\frac{\partial f}{\partial x}\right)^2$+$\left(\frac{\partial f}{\partial y}\right)^2$

5) $z = f(x, y)$, $x = u + v$, $y = u - v$, $then$:

show $that$ $z_u z_v = \left(\frac{\partial f}{\partial x}\right)^2$-$\left(\frac{\partial f}{\partial y}\right)^2$

6) Suppose that $z = f(x - iy) + g(x + iy)$.

Prove that $z_{xx} + z_{yy} = 0$,where $\sqrt{-1} = i$