Lecture Notes in Foundations of Mathematics

Department of Mathematics

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Chapter

1 Logic and Proofs

Section 1.1: Propositions and Connectives

Definition 1.1.1

A **proposition P** is a sentence which is either true **T** or false **F**. That is, the truth values of propositions are **T** or **F**.

Example 1.1.1

Consider the following sentences:

- Propositions:

  a) $\frac{1}{2}$ is a rational number. [**T**].

  b) $2 + 4 = 1$. [**F**].

- Not propositions:

  c) How are you doing? [not a proposition].

  d) $x^2 = 36$. [where is x coming from?].

  e) This sentence is false. [depends on the given sentence!].

The previous propositions studied in *a* and *b* are called **simple** propositions. **Compound** propositions can be formed by **connectives** with simple propositions. For example,

Compound proposition: $1 + 2 = 5$ "and" the sun is made of an orange.

Definition 1.1.2

Let **P** and **Q** be two propositions. Then,

1. the **conjunction** of **P** and **Q**, denoted by $\mathbf{P} \wedge \mathbf{Q}$, is the proposition "**P** and **Q**". $\mathbf{P} \wedge \mathbf{Q}$ is true exactly when both **P** and **Q** are true.

2. the **disjunction** of $\mathbf{P}$ and $\mathbf{Q}$, denoted by $\mathbf{P} \vee \mathbf{Q}$, is the proposition "$\mathbf{P}$ or $\mathbf{Q}$". $\mathbf{P} \vee \mathbf{Q}$ is true exactly when at least one of $\mathbf{P}$ or $\mathbf{Q}$ is true.
3. the **negation** of $\mathbf{P}$, denoted by $\sim \mathbf{P}$, is the proposition "not $\mathbf{P}$". $\sim \mathbf{P}$ is true exactly when $\mathbf{P}$ is false.

Example 1.1.2

Let $\mathbf{P}$ be "Kuwait is an island" and let $\mathbf{Q}$ be "Sea water contains salt". Discuss $\mathbf{P} \wedge \mathbf{Q}$, $\mathbf{P} \vee \mathbf{Q}$, and $\sim \mathbf{P}$.

Solution:

It is clear the $\mathbf{P}$ is false and $\mathbf{Q}$ is true. Thus,

1. $\mathbf{P} \wedge \mathbf{Q}$: Kuwait is an island and sea water contains salt. [**F**].
2. $\mathbf{P} \vee \mathbf{Q}$: Kuwait is an island or sea water contains salt. [**T**].
3. $\sim \mathbf{P}$: It is not the case that Kuwait is an island. [**T**].

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{P} \wedge \mathbf{Q}$	$\mathbf{P} \vee \mathbf{Q}$	$\sim \mathbf{P}$	$\sim \mathbf{Q}$
T	**T**	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**	**T**
F	**T**	**F**	**T**	**T**	**F**
F	**F**	**F**	**F**	**T**	**T**

Definition 1.1.3

A **propositional form** is an expression involving finitely many propositions connected by connectives such as $\wedge$, $\vee$, and $\sim$.

Example 1.1.3

Let $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ be propositions. Write down the truth table of the propositional form $\big((\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \vee (\sim \mathbf{R}))\big)$.

Solution:

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{R}$	$\sim\mathbf{R}$	$\mathbf{P}\wedge\mathbf{Q}$	$\mathbf{P}\vee(\sim\mathbf{R})$	$\big((\mathbf{P}\wedge\mathbf{Q})\vee(\mathbf{P}\vee(\sim\mathbf{R}))\big)$
T	**T**	**T**	**F**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**	**T**
F	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**T**
F	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**T**	**F**	**T**	**T**

Definition 1.1.4

Two propositional forms $\mathbf{P}$ and $\mathbf{Q}$ are called **equivalent** if and only if their truth tables are identical. In that case, we write $\mathbf{P}\equiv\mathbf{Q}$.

Definition 1.1.5

A **denial** of a proposition $\mathbf{P}$ is any proposition equivalent to $\sim\mathbf{P}$.

A proposition $\mathbf{P}$ has only one negation "$\sim\mathbf{P}$", but it has many denials. For instance, $\sim\mathbf{P}$, $\sim\sim\sim\mathbf{P}$, and $\sim\sim\sim\sim\sim\mathbf{P}$ are all examples of denials. Note that $\sim(\sim\mathbf{P})$ is simply $\mathbf{P}$.

Example 1.1.4

Let $\mathbf{P}$ be "π is an irrational number". Find the negation of $\mathbf{P}$, and give some examples of denials of $\mathbf{P}$.

Solution:

- negation $\sim\mathbf{P}$: It is not the case that π is irrational.
- denials of $\mathbf{P}$: a. π is rational. b. π is the quotient of two integers r/s. c. π has a finite decimal expansion.

Note that since $\mathbf{P}$ is true, all of its denials are false.

Definition 1.1.6

A propositional form is called a **tautology** if it is true for all possible truth values of its components. It is called a **contradiction** if it is the negation of a tautology.

Example 1.1.5

Show that $\big((\mathbf{P} \vee \mathbf{Q}) \vee ((\sim \mathbf{P}) \wedge (\sim \mathbf{Q}))\big)$ is a tautology for any propositions $\mathbf{P}$ and $\mathbf{Q}$.

Solution:

$\mathbf{P}$	$\mathbf{Q}$	$\sim \mathbf{P}$	$\sim \mathbf{Q}$	$\mathbf{P} \vee \mathbf{Q}$	$(\sim \mathbf{P}) \wedge (\sim \mathbf{Q})$	$\big((\mathbf{P} \vee \mathbf{Q}) \vee ((\sim \mathbf{P}) \wedge (\sim \mathbf{Q}))\big)$
T	**T**	**F**	**F**	**T**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**	**T**
F	**T**	**T**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**	**T**	**T**

Moreover, it can be seen that the negation of $\big((\mathbf{P} \vee \mathbf{Q}) \vee ((\sim \mathbf{P}) \wedge (\sim \mathbf{Q}))\big)$ is a contradiction.

Remark 1.1.1

The negation of a tautology is a contradiction, and the negation of a contradiction is a tautology.

Section 1.2: Conditionals and Biconditionals

Definition 1.2.1

Given two propositions $\mathbf{P}$ and $\mathbf{Q}$, the conditional sentence $\mathbf{P} \Rightarrow \mathbf{Q}$ (reads "$\mathbf{P}$ implies $\mathbf{Q}$") is the proposition "if $\mathbf{P}$, then $\mathbf{Q}$". In that case, $\mathbf{P}$ is called **antecedent** and $\mathbf{Q}$ is called **consequent**.

Remark 1.2.1

The proposition $\mathbf{P} \Rightarrow \mathbf{Q}$ is true whenever $\mathbf{P}$ is false or $\mathbf{Q}$ is true. In general, $\mathbf{P} \Rightarrow \mathbf{Q}$ is equivalent to $(\sim \mathbf{P}) \vee \mathbf{Q}$.

Example 1.2.1

Consider the following propositions:

a) if "x is an odd integer", then "$x+1$ is an even integer". [**T**].

b) if "$2+1=0$", then "$1+1=0$". [**T**].

c) if "$1-1=0$", then "$2+9=1$". [**F**].

Definition 1.2.2

For propositions $\mathbf{P}$ and $\mathbf{Q}$, the **converse** of $\mathbf{P} \Rightarrow \mathbf{Q}$ is $\mathbf{Q} \Rightarrow \mathbf{P}$, and the **contrapositive** of $\mathbf{P} \Rightarrow \mathbf{Q}$ is $(\sim \mathbf{Q}) \Rightarrow (\sim \mathbf{P})$.

Theorem 1.2.1

For any propositions $\mathbf{P}$ and $\mathbf{Q}$, we have
(i) $\mathbf{P} \Rightarrow \mathbf{Q}$ is equivalent to $(\sim \mathbf{Q}) \Rightarrow (\sim \mathbf{P})$, and (ii) $\mathbf{P} \Rightarrow \mathbf{Q}$ is not equivalent to $\mathbf{Q} \Rightarrow \mathbf{P}$.

Proof:

We prove both results in the following truth table.

$\mathbf{P}$	$\mathbf{Q}$	$\sim\mathbf{P}$	$\sim\mathbf{Q}$	$\mathbf{P}\Rightarrow\mathbf{Q}$	$\mathbf{Q}\Rightarrow\mathbf{P}$	$\sim\mathbf{Q}\Rightarrow\sim\mathbf{P}$
T	**T**	**F**	**F**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

Definition 1.2.3

Let $\mathbf{P}$ and $\mathbf{Q}$ be two propositions. The **biconditional** sentence $\mathbf{P} \Leftrightarrow \mathbf{Q}$ is "$\mathbf{P}$ if and only if (iff.) $\mathbf{Q}$". $\mathbf{P} \Leftrightarrow \mathbf{Q}$ is true exactly when both $\mathbf{P}$ and $\mathbf{Q}$ have the same truth value.

Remark 1.2.2

The following phrases are translated as $\mathbf{P} \Rightarrow \mathbf{Q}$ for any propositions $\mathbf{P}$ and $\mathbf{Q}$:

• if $\mathbf{P}$, then $\mathbf{Q}$.	• if $a > 5$, then $a > 3$.
• $\mathbf{P}$ implies $\mathbf{Q}$.	• $a > 5$ implies $a > 3$.
• $\mathbf{P}$ is sufficient for $\mathbf{Q}$.	• $a > 5$ is sufficient for $a > 3$.
• $\mathbf{P}$ only if $\mathbf{Q}$.	• $a > 5$ only if $a > 3$
• $\mathbf{Q}$, if $\mathbf{P}$.	• $a > 3$, if $a > 5$.
• $\mathbf{Q}$ whenever $\mathbf{P}$.	• $a > 3$ whenever $a > 5$.
• $\mathbf{Q}$ is necessary for $\mathbf{P}$.	• $a > 3$ is necessary for $a > 5$.
• $\mathbf{Q}$, when $\mathbf{P}$.	• $a > 3$, when $a > 5$.

Remark 1.2.3

Moreover, the following phrases are translated as $\mathbf{P} \Leftrightarrow \mathbf{Q}$ for any propositions $\mathbf{P}$ and $\mathbf{Q}$:

• $\mathbf{P}$ if and only if $\mathbf{Q}$.	• $\|x\| = 2$ iff $x^2 = 4$.
• $\mathbf{P}$ if, but only if, $\mathbf{Q}$.	• $\|x\| = 2$ if, but only if, $x^2 = 4$.
• $\mathbf{P}$ is equivalent to $\mathbf{Q}$.	• $\|x\| = 2$ is equivalent to $x^2 = 4$.
• $\mathbf{P}$ is necessary and sufficient for $\mathbf{Q}$.	• $\|x\| = 2$ is necessary and sufficient for $x^2 = 4$.

Theorem 1.2.2

Let $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ be propositions. Then,

a. $\mathbf{P} \Rightarrow \mathbf{Q} \equiv (\sim \mathbf{P}) \vee \mathbf{Q}$.

b. $\mathbf{P} \Leftrightarrow \mathbf{Q} \equiv (\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})$.

c. $\sim (\mathbf{P} \wedge \mathbf{Q}) \equiv (\sim \mathbf{P}) \vee (\sim \mathbf{Q})$.

d. $\sim (\mathbf{P} \vee \mathbf{Q}) \equiv (\sim \mathbf{P}) \wedge (\sim \mathbf{Q})$.

e. $\sim (\mathbf{P} \Rightarrow \mathbf{Q}) \equiv \mathbf{P} \wedge (\sim \mathbf{Q})$.

f. $\sim (\mathbf{P} \wedge \mathbf{Q}) \equiv \mathbf{P} \Rightarrow (\sim \mathbf{Q})$.

g. $\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R}) \equiv (\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R})$.

h. $\mathbf{P} \vee (\mathbf{Q} \wedge \mathbf{R}) \equiv (\mathbf{P} \vee \mathbf{Q}) \wedge (\mathbf{P} \vee \mathbf{R})$.

Proof:

b.

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{P} \Leftrightarrow \mathbf{Q}$	$\mathbf{P} \Rightarrow \mathbf{Q}$	$\mathbf{Q} \Rightarrow \mathbf{P}$	$(\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})$
T	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**F**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**

g.

$\mathbf{P}$	$\mathbf{Q}$	$\mathbf{R}$	$\mathbf{Q} \vee \mathbf{R}$	$\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R})$	$\mathbf{P} \wedge \mathbf{Q}$	$\mathbf{P} \wedge \mathbf{R}$	$(\mathbf{P} \vee \mathbf{Q}) \vee (\mathbf{P} \vee \mathbf{R})$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**