Section 1.3: Quantifiers

⋆ NOTATIONS:

- $\mathbb{N} = \{1, 2, 3, \cdots\}$ is the set of **natural numbers**.
- $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is the set of **integer numbers**.
- $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}$ and $q \neq 0\}$ is the set of **rational numbers**.
- $\mathbb{R}$ is the set of **real numbers**.

The sentence $x \geq 5$ is not a proposition, unless we assign a value to x. It is an open sentence. In general, an open sentence with n variables is denoted by $P(x_1, x_2, \cdots, x_n)$. For example, the open sentence $P(x_1, x_2, x_3)$: "x_1 equals to $x_2 + x_3$" is an open sentence. On the other hand, $P(7, 3, 4)$ and $P(7, 2, 3)$ are propositions with true and false values, respectively.

Definition 1.3.1

The set of objects for which an open sentence is true is called the **truth set**, and is denoted by $\mathcal{T}$.

On the other hand, the set from where the objects can be taken from is called the **universe**, and is denoted by $\mathcal{U}$. In particular, two open sentences are said to be equivalent for a particular universe if and only if their truth sets are equal.

Example 1.3.1

Let $\mathcal{U} = \mathbb{N}$. Then, $P(x) : x + 3 > 7$ is equivalent to $Q(x) : x > 4$, since $\mathcal{T} = \{5, 6, 7, \cdots\}$ for both P and Q.

Also, $P(x) : x^2 = 4$ is equivalent to $Q(x) : x = 2$. However, if $\mathcal{U}$ was the set of all integers, then $P(x) : x^2 = 4$ with truth set $\{-2, 2\}$ is not equivalent to $Q(x) : x = 2$ with truth set $\{2\}$.

Definition 1.3.2

Let $\mathbf{P}(x)$ be an open sentence with variable $x \in \mathcal{U}$. Then,

a) The sentence "$(\forall x)\mathbf{P}(x)$" reads as "for all x, $\mathbf{P}(x)$". It is true iff $\mathcal{T} = \mathcal{U}$ for $\mathbf{P}(x)$. "$\forall$" is called the **universal quantifiers**.

b) The sentence "$(\exists x)\mathbf{P}(x)$" reads as "there exists x such that $\mathbf{P}(x)$". It is true iff $\mathcal{T} \neq \emptyset$ (the empty set). "$\exists$ is called the **existential quantifiers**.

c) The sentence "$(\exists! x)\mathbf{P}(x)$" reads as "there exists a unique x such that $\mathbf{P}(x)$". It is true iff $\mathcal{T}$ contains only one element. "$\exists!$ is called the **unique existential quantifiers**.

Example 1.3.2

Let $\mathcal{U} = \mathbb{R}$. Decide the truth value and the truth set for each of the following.

Solution:

Consider the following table where we different sentences along with its truth value as true or false and the corresponding truth set.

sentence	**T** or **F**	$\mathcal{T}$
a. $(\forall x)(x \geq 3)$	**F**	$[3, \infty)$.
b. $(\forall x)(\lvert x\rvert > 0)$	**F**	$\mathbb{R}\backslash\{0\}$.
c. $(\forall x)(x - 1 < x)$	**T**	$\mathbb{R}$.
d. $(\exists x)(x \geq 3)$	**T**	$[3, \infty)$.
e. $(\exists! x)(\lvert x\rvert = 0)$	**T**	$\{0\}$.
f. $(\exists! x)(\lvert x\rvert = 2)$	**F**	$\{-2, 2\}$.
g. $(\exists x)(x^2 = -4)$	**F**	$\emptyset$.
h. $(\exists x)(\exists y)(2x + y = 0 \wedge x - y = 1)$	**T**	$\{x = \frac{1}{3}, y = -\frac{2}{3}\}$.
i. $(\exists! x)(\exists! y)(2x + y = 0 \vee x - y = 1)$	**F**	$(x, y) \in \Big\{(0,0), (1,0), (3,2), \cdots\Big\}$.
j. $(\forall x)(\forall y)(x^2 + y^2 > 0)$	**F**	$\mathbb{R}^2\backslash(0,0)$.

Definition 1.3.3

Two quantified sentences are equivalent for a particular universe $\mathcal{U}$ iff they have the same truth set in $\mathcal{U}$. Two quantified sentences are equivalent iff they are equivalent in every universe.

For instance, $(\forall x)(\mathbf{P}(x) \wedge \mathbf{Q}(x))$ is equivalent to $(\forall x)(\mathbf{Q}(x) \wedge \mathbf{P}(x))$ and $(\forall x)[\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)]$ is equivalent to $(\forall x)[\sim \mathbf{Q}(x) \Rightarrow \sim \mathbf{P}(x)]$.

Theorem 1.3.1

Let $\mathbf{P}(x)$ be an open sentence with a variable $x \in \mathcal{U}$ for some $\mathcal{U}$. Then,

a. $\sim (\forall x)[\mathbf{P}(x)]$ is equivalent to $(\exists x)[\sim \mathbf{P}(x)]$.

b. $\sim (\exists x)[\mathbf{P}(x)]$ is equivalent to $(\forall x)[\sim \mathbf{P}(x)]$.

Proof:

(a.) The sentence $\sim (\forall x)[\mathbf{P}(x)]$ is true iff $(\forall x)[\mathbf{P}(x)]$ is false iff the truth set for $\mathbf{P}(x)$ is not the entire universe, i.e. $\mathcal{T} \neq \mathcal{U}$ iff there exists an $x \in \mathcal{U}$ such that $\mathbf{P}(x)$ is false iff $(\exists x)[\sim \mathbf{P}(x)]$ is true.

(b.) The sentence $\sim (\exists x)[\mathbf{P}(x)]$ is true iff $(\exists x)[\mathbf{P}(x)]$ is false iff the truth set of $\mathbf{P}(x)$ is empty iff $(\forall x)[\sim \mathbf{P}(x)]$ is true.

Remark 1.3.1

Let $\mathbf{P}(x)$ be an open sentence with a variable $x \in \mathcal{U}$ for some $\mathcal{U}$. Then,

$$(\exists! x)\mathbf{P}(x) \;\equiv\; (\exists x)\Big[\mathbf{P}(x) \wedge (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big].$$

Example 1.3.3

Find a denial (or the negation) for "$(\forall x)[\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)]$".

Solution:

Using Theorem 1.3.1 and Theorem 1.2.2 (part e), we conclude

$$\sim (\forall x)[\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)] \equiv (\exists x)[\sim (\mathbf{P}(x) \Rightarrow \mathbf{Q}(x))] \equiv (\exists x)[\mathbf{P}(x) \wedge (\sim \mathbf{Q}(x))].$$

Example 1.3.4

Find a denial (or the negation) for "$(\exists! x)\mathbf{P}(x)$".

Solution:

Using Remark 1.3.1 and Theorem 1.2.2, we conclude

$$\begin{aligned}
\sim (\exists! x)\mathbf{P}(x) &\equiv \sim (\exists x)\Big[\mathbf{P}(x) \wedge (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big] \\
&\equiv (\forall x)\Big[\sim \Big(\mathbf{P}(x) \wedge (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big)\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee \sim (\forall y)[\mathbf{P}(y) \Rightarrow x = y]\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee (\exists y) \sim [\mathbf{P}(y) \Rightarrow x = y]\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee (\exists y)[\mathbf{P}(y) \wedge \sim (x = y)]\Big] \\
&\equiv (\forall x)\Big[\sim \mathbf{P}(x) \vee (\exists y)[\mathbf{P}(y) \wedge x \neq y]\Big]
\end{aligned}$$

Example 1.3.5

Find a denial (or the negation) for

$$(\forall z)(\exists x)(\exists y)\Big[\Big((x > z) \wedge (y > z)\Big) \wedge \sim (\exists w)\Big(x + y < w < xz\Big)\Big]. \tag{1.3.1}$$

Solution:

Using Theorem1.3.1 and Theorem 1.2.2, we conclude

$$\begin{aligned}
\sim \text{Equation}(1.3.5) &\equiv \sim (\forall z)(\exists x)(\exists y)\Big[\Big((x > z) \wedge (y > z)\Big) \wedge \sim (\exists w)\Big(x + y < w < xz\Big)\Big] \\
&\equiv (\exists z)(\forall x)(\forall y) \sim \Big[\Big((x > z) \wedge (y > z)\Big) \wedge \sim (\exists w)\Big(x + y < w < xz\Big)\Big] \\
&\equiv (\exists z)(\forall x)(\forall y)\Big[\Big((x > z) \wedge (y > z)\Big) \Rightarrow \sim\sim (\exists w)\Big(x + y < w < xz\Big)\Big] \\
&\equiv (\exists z)(\forall x)(\forall y)\Big[\Big((x > z) \wedge (y > z)\Big) \Rightarrow (\exists w)\Big(x + y < w < xz\Big)\Big].
\end{aligned}$$

Example 1.3.6

Let $\mathcal{U} = \mathbb{R}$. Decide the truth value and the truth set for each of the following.

Solution:

sentence	**T** or **F**	$\mathcal{T}$
a. $(\forall y)(\exists x)[x + y = 0]$	**T**	for any y, $x = -y$ is a solution.
b. $(\exists x)(\forall y)[x + y = 0]$	**F**	given $x = 0$ not all $y \in \mathbb{R}$ is a solution.
c. $(\exists x)(\exists y)[x^2 + y^2 = 10]$	**T**	for $x \in \mathbb{R}$ there is $y = \sqrt{10 - x^2} \in \mathbb{R}$.
d. $(\forall y)(\exists x)(\forall z)[xy = xz]$	**T**	for any $y \in \mathbb{R}$, $x = 0$ for any $z \in \mathbb{R}$.
e. $(\forall y)(\exists! x)[x = y^2]$	**T**	for any $y \in \mathbb{R}$, $x = y^2$ is a solution.

Section 1.4: Mathematical Proofs

Definition 1.4.1

A **proof** is a justification of the truth of a given statement called theorem, proposition, claim, or lemma.

Remark 1.4.1

Tools of proofs: We may use any of the following:

- Axioms: Initial statements which are assumed to be true.
- Theorems: Some previously proved statement can be use.
- Assumptions: Assumed fact about the problem at hand.
- Tautologies: Examples follow:

a. $\mathbf{P} \vee (\sim \mathbf{P})$ .... (Excluded Middle).

b. $(\mathbf{P} \Rightarrow \mathbf{Q}) \Leftrightarrow (\sim \mathbf{Q} \Rightarrow \sim \mathbf{P})$ .... (Contrapositive).

c. $\left.\begin{array}{l} \mathbf{P} \vee (\mathbf{Q} \vee \mathbf{R}) \Leftrightarrow (\mathbf{P} \vee \mathbf{Q}) \vee \mathbf{R} \\ \mathbf{P} \wedge (\mathbf{Q} \wedge \mathbf{R}) \Leftrightarrow (\mathbf{P} \wedge \mathbf{Q}) \wedge \mathbf{R} \end{array}\right\}$ .... (Associativity).

d. $\left.\begin{array}{l} \mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R}) \Leftrightarrow (\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R}) \\ \mathbf{P} \vee (\mathbf{Q} \wedge \mathbf{R}) \Leftrightarrow (\mathbf{P} \vee \mathbf{Q}) \wedge (\mathbf{P} \vee \mathbf{R}) \end{array}\right\}$ .... (Distributivity).

e. $(\mathbf{P} \Leftrightarrow \mathbf{Q}) \Leftrightarrow [(\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})]$ .... (Biconditional).

f. $\sim (\mathbf{P} \Rightarrow \mathbf{Q}) \Leftrightarrow (\mathbf{P} \wedge \sim \mathbf{Q})$ .... (Denial of Implication).

g. $\left.\begin{array}{l} \sim (\mathbf{P} \wedge \mathbf{Q}) \Leftrightarrow (\sim \mathbf{P} \vee \sim \mathbf{Q}) \\ \sim (\mathbf{P} \vee \mathbf{Q}) \Leftrightarrow (\sim \mathbf{P} \wedge \sim \mathbf{Q}) \end{array}\right\}$ .... (De Morgan's Laws).

h. $\mathbf{P} \Leftrightarrow [\sim \mathbf{P} \Rightarrow (\mathbf{Q} \wedge \sim \mathbf{Q})]$ .... (Contradiction).

i. $[(\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{R})] \Leftrightarrow (\mathbf{P} \Rightarrow \mathbf{R})$ .... (Transitivity).

j. $[\mathbf{P} \wedge (\mathbf{P} \Rightarrow \mathbf{Q})] \Rightarrow \mathbf{Q}$ .... (Modus Ponens).

In what follows, we consdier different types of proof.

1.4.1 Type 1: Direct Proof

Direct proof $\mathbf{P} \Rightarrow \mathbf{Q}$: Assume $\mathbf{P}$, then $\cdots\ \cdots$. Therefore, $\mathbf{Q}$.

Example 1.4.1

Let n be an integer. Show that if n is odd, then $n+1$ is even.

Solution:

Assume that $n = 2k+1$ for some integer k. Then, $n+1 = (2k+1)+1$. That is $n+1 = 2k+2 = 2(k+1)$. Therefore, $n+1$ is even.

Example 1.4.2

Assume that $\sin(x)$ is an odd funtion, i.e. $\sin(-x) = -\sin(x)$. Show that $f(x) = \sin^2(x)$ for any $x \in \mathbb{R}$ is an even function, i.e. $f(-x) = f(x)$.

Solution:

$f(-x) = (\sin(-x))^2 = (-\sin(x))^2 = \sin(x) = f(x)$. Therefore, $f(x)$ is an even function.

Theorem 1.4.1

Suppose that a, b, and c are integers. If a divides b and b divides c, then a divides c.

Proof:

Since a divides b ($a \mid b$), then there is an integer k such that $b = ka$. Also, since $b \mid c$ there is an integer h such that $c = hb$. Thus, $c = hb = h(ka) = (hk)a$, and therefore $a \mid c$.

Theorem 1.4.2

Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $a \mid c$, then $a \mid b \pm c$.

Proof:

Since $a \mid b$, $\exists k \in \mathbb{Z}$ such that $b = ka$, and since $a \mid c$, $\exists h \in \mathbb{Z}$ such that $c = ha$. Thus,

$$b \pm c = ka \pm ha = (k \pm h)a.$$

Therefore, $a \mid b \pm c$.

1.4.2 Type 2: Proof By Contradiction

Contradiction to proof **P**: Suppose $\sim \mathbf{P}$, then $\cdots\cdots$. Thus **Q**. Then, $\cdots\cdots$. Therefore, $\sim \mathbf{Q}$, contradiction.

This technique uses the tautology $\mathbf{P} \Leftrightarrow [\sim \mathbf{P} \Rightarrow (\mathbf{Q} \wedge \sim \mathbf{Q})]$.

Example 1.4.3

The equation $x^3 + x - 1 = 0$ has at most one real root.

Solution:

Let $f(x) = x^3 + x - 1$. Suppose that $f(x)$ has two real roots a and b, then $f(a) = f(b) = 0$. f is continuouse on $[a, b]$ and is differentiable on (a, b) since it is a polynomial. Then, by Rolle's Theorem, there is a $c \in (a, b)$ such that $f'(c) = 0$. But $f'(c) = 3c^2 + 1 \neq 0$ for all $c \in \mathbb{R}$. This is a contradiction. Therefore, f has at most one real root.

Remark 1.4.2

- Any square integer has an even number of 2's as prime factors.
- All natural number greater than 1 has a prime divisor $q > 1$.

Example 1.4.4

Prove that $\sqrt{2}$ is an irrational number.

Solution:

Recall the fact that any square integer number has an even number of 2's as prime factors. Suppose that $\sqrt{2}$ is rational number. Then, $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Thus, $2 = \frac{p^2}{q^2}$ or $p^2 = 2q^2$. Since p^2 and q^2 are both square numbers, p^2 contains an even number of 2's as prime factors (might be 0 times for odd numbers) and q^2 contains an even number of 2's as prime factors. But then $2q^2$ has an odd number of 2's as prime factors and thus p^2 has an odd number of 2's as prime factors because $p^2 = 2q^2$. This is a contradiction. Thus, $\sqrt{2}$ is an irrational number.