Chapter

2 Set Theory

Section 2.1: Basic Notations of Set Theory

Definition 2.1.1

A **set** is a collection of objects called elements. Sets are usually denoted by capital letters $A, B, C, \cdots$ while elements are usually denoted by small letters $a, b, c, \cdots$.

- If a is an element of a set A, then we write $a \in A$. Otherwise, we write $a \notin A$.
- The empty set $\phi := \{x : x \neq x\}$. That is, ϕ is a set with no elements.
- A set B is a **subset** of A, denoted by $B \subseteq A$, if and only if every elements of B is also an element of A. That is, $\forall b \in B \Rightarrow b \in A$.
- A set B is called a **proper subset** of set A, if $B \subseteq A$ and $B \neq \phi$, but $B \neq A$. In this case, we write $B \subset A$.
- Two subsets A and B are equal , denoted by $A = B$, if and only of $A \subseteq B$ and $B \subseteq A$.
- If a set A contains n elements, we say that $|A| = n$.

Theorem 2.1.1

For any sets A, B, and C, we have:

1) $\phi \subseteq A$,
2) $A \subseteq A$, and
3) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof:

The first two results are trivial so we leave those. For part 3) let a be any element of A. Since $A \subseteq B$, $a \in B$. But since $B \subseteq C$, $a \in C$. Thus, if $a \in A \Rightarrow a \in C$. Thus, $A \subseteq C$.

Definition 2.1.2

Let A be a set. The **power set** of A is the set whose elements are all the subsets of A and is denoted by $\mathcal{P}(A)$. Thus,

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

Example 2.1.1

Let $A = \{a, b, c\}$. Find $\mathcal{P}(A)$.

Solution:

$\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$.

Remark 2.1.1

Let A be any given set. Then,

a. Theorem: If $|\,A\,| = n$, then $|\,\mathcal{P}(A)\,| = 2^n$.
b. $A \not\subseteq \mathcal{P}(A)$, but $A \in \mathcal{P}(A)$.

Example 2.1.2

Let $A = \{1, \{1, 3\}, \{2, 3\}\}$. Find $\mathcal{P}(A)$.

Solution:

$$\mathcal{P}(A) = \{\,\phi, \{1\}, \{\,\{\,1, 3\,\}\,\}, \{\,\{\,2, 3\,\}\,\}, \{\,1, \{\,1, 3\,\}\,\}, \{\,1, \{\,2, 3\,\}\,\}, \{\,\{\,1, 3\,\}, \{\,2, 3\,\}\,\}, A\,\}.$$

Note that, $1 \in A$, while $2 \notin A$ and $3 \notin A$. Also, $\{1\} \notin A$ where $\{2, 3\} \in A$ and $\{\{2, 3\}\} \subseteq A$ hence $\{\{2, 3\}\} \in \mathcal{P}(A)$. Moreover, $1 \notin \mathcal{P}(A)$, $\{1\} \in \mathcal{P}(A)$, and $\{\{1\}\} \subseteq \mathcal{P}(A)$. Also, $\phi \subseteq A$, $\phi \in \mathcal{P}(A)$ and $\{\phi\} \subseteq \mathcal{P}(A)$. Finally, $\{1, 3\} \notin \mathcal{P}(A)$, but $\{\{1, 3\}\} \in \mathcal{P}(A)$ and $\{\{\{1, 3\}\}\} \subseteq \mathcal{P}(A)$.

Theorem 2.1.2

Let A and B be two sets. Then, $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof:

„ $\Rightarrow$ ": Assume that $A \subseteq B$. Let $X \in \mathcal{P}(A)$. Then, $X \subseteq A \subseteq B$. That is, $X \in \mathcal{P}(B)$. Thus, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

„ $\Leftarrow$ ": Assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have $A \in \mathcal{P}(B) \Rightarrow A \subseteq B$.

Exercise 2.1.1

Let $A = \{\, 9^n : n \in \mathbb{Z} \,\}$ and $B = \{\, 3^n : n \in \mathbb{Z} \,\}$. Show that $A \subsetneqq B$.

Exercise 2.1.2

Let $A = \{\, 9^n : n \in \mathbb{Q} \,\}$ and $B = \{\, 3^n : n \in \mathbb{Q} \,\}$. Show that $A = B$.

Exercise 2.1.3

Find $\mathcal{P}(\phi)$, $\mathcal{P}(\mathcal{P}(\phi))$, and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\phi)))$.

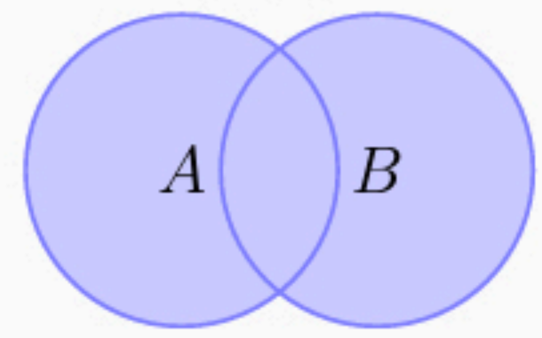
Section 2.2: Set Operations

Definition 2.2.1

Let A and B be two sets. Then,

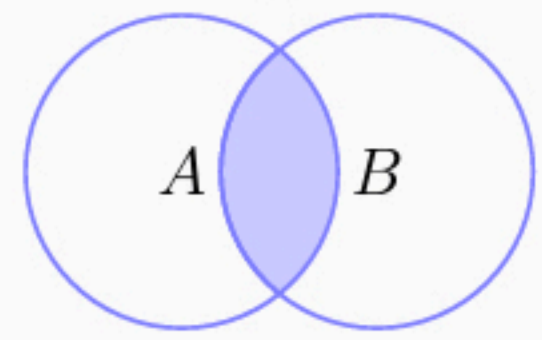
1. **Union**: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
   What is the meaning of $x \notin A \cup B$?

$A \cup B$



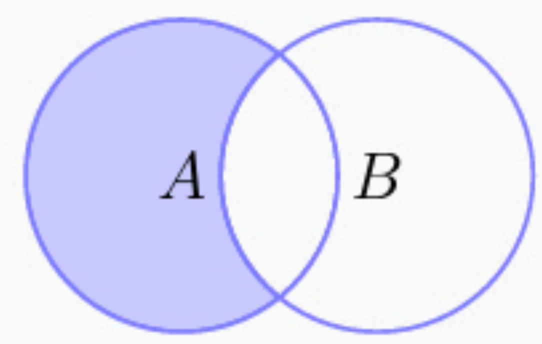
2. **intersection**: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
   What is the meaning of $x \notin A \cap B$?

$A \cap B$



3. **Difference**: $A - B = \{x : x \in A \text{ and } x \notin B\}$.
   What is the meaning of $x \notin A - B$?

$A - B$

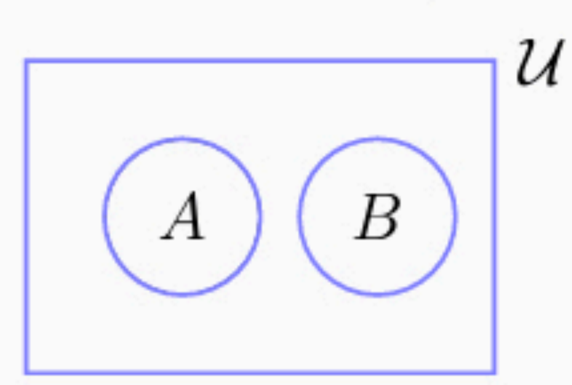


4. **Complement**: If $\mathcal{U}$ is the universal, then
   $\widetilde{A} = \{x : x \notin A\} = \{x : x \in \mathcal{U} - A\}$.

$\widetilde{A} = \mathcal{U} - A$

5. **Disjoint**: A and B are called **disjoint** if $A \cap B = \phi$.

$A \cap B = \phi$



Theorem 2.2.1

Let A, B, and C be sets. Then,

1. $A \subseteq A \cup B$.
2. $A \cap B \subseteq A$.
3. $A \cap \phi = \phi$.
4. $A \cup \phi = A$.

5. $A \cap A = A$.
6. $A \cup A = A$.
7. $A \cup B = B \cup A$.
8. $A \cap B = B \cap A$.
9. $A - \phi = A$.
10. $\phi - A = \phi$.
11. $A \cup (B \cup C) = (A \cup B) \cup C$.
12. $A \cap (B \cap C) = (A \cap B) \cap C$.
13. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
14. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
15. $A \subseteq B$ if and only if $A \cup B = B$.
16. $A \subseteq B$ if and only if $A \cap B = A$.
17. if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.
18. if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

Proof:

Proof of (13): Using the fact "$\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R}) = (\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R})$" as follows.

$$\begin{aligned} x \in A \cap (B \cup C) \quad & \text{iff} \quad x \in A \text{ and } x \in B \cup C \\ & \text{iff} \quad x \in A \text{ and } (x \in B \text{ or } x \in C) \\ & \text{iff} \quad (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ & \text{iff} \quad x \in A \cap B \text{ or } x \in A \cap C \\ & \text{iff} \quad x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Proof of (15): „ $\Rightarrow$ ": Assume that $A \subseteq B$. By part (1), $B \subseteq A \cup B$ so we only show that $A \cup B \subseteq B$. Let $x \in A \cup B$, then $x \in A \subseteq B$ or $x \in B$. In both cases, $x \in B$. Thus, $A \cup B \subseteq B$. Therefore, $B = A \cup B$.

„ $\Leftarrow$ ": Assume that $A \cup B = B$. By part (1) $A \subseteq A \cup B = B$. Thus, $A \subseteq B$.

Proof of (18): Assume that $A \subseteq B$. Let $x \in A \cap C$, then $x \in A \subseteq B$ and $x \in C$. Thus, $x \in B$ and $x \in C$, which implies that $x \in B \cap C$. Therefore, $A \cap C \subseteq B \cap C$.

Theorem 2.2.2

Let A and B be two subsets of the universe $\mathcal{U}$. Then:

1. $\tilde{\tilde{A}} = A$.
2. $A \cup \tilde{A} = \mathcal{U}$.
3. $A \cap \tilde{A} = \phi$.
4. $A - B = A \cap \tilde{B}$.
5. $A \subseteq B$ iff $\tilde{B} \subseteq \tilde{A}$.
6. $A \cap B = \phi$ iff $A \subseteq \tilde{B}$.
7. $\left.\begin{array}{l} \text{a. } \widetilde{A \cup B} = \tilde{A} \cap \tilde{B}. \\ \text{b. } \widetilde{A \cap B} = \tilde{A} \cup \tilde{B}. \end{array}\right\}$ .... (De Morgan's Laws).

Proof:

Proof of (2): If $x \in A \cup \tilde{A}$ then $x \in A \subseteq \mathcal{U}$ or $x \in \tilde{A} = \mathcal{U} - A$. In either cases, $x \in \mathcal{U}$. Thus, $A \cup \tilde{A} \subseteq \mathcal{U}$.

Assume now that $x \in \mathcal{U}$. Thus, $x \in A$ or $x \in \mathcal{U} - A = \tilde{A}$ which implies $x \in A \cup \tilde{A}$. Thus $\mathcal{U} \subseteq A \cup \tilde{A}$. Therefore, $\mathcal{U} = A \cup \tilde{A}$.

Proof of (5): Using a contrapositive proof as follows:

$$\begin{array}{rll} A \subseteq B & \text{iff} & (\forall x)(x \in A \Rightarrow x \in B) \\ & \text{iff} & (\forall x)(x \notin B \Rightarrow x \notin A) \\ & \text{iff} & (\forall x)(x \in \tilde{B} \Rightarrow x \in \tilde{A}) \\ & \text{iff} & \tilde{B} \subseteq \tilde{A}. \end{array}$$

Proof of (7.b): Recall that $\sim(\mathbf{P} \wedge \mathbf{Q}) = \sim \mathbf{P} \vee \sim \mathbf{Q}$:

$$\begin{array}{rll} x \in \widetilde{A \cap B} & \text{iff} & x \notin A \cap B \\ & \text{iff} & \sim (x \in A \text{ and } x \in B) \\ & \text{iff} & x \notin A \text{ or } x \notin B \\ & \text{iff} & x \in \tilde{A} \text{ or } x \in \tilde{B} \\ & \text{iff} & x \in \tilde{A} \cup \tilde{B}. \end{array}$$

Example 2.2.1

Let $\mathcal{U} = \{1,2,3,4,5,6,7,8\}$ be the universe and let $A = \{1,5,7\}$, $B = \{2,5,8\}$, and $C = \{3,4,5,6,7\}$ Answer Each of the following:

1. $A \cap B = \{5\}$.
2. $B \cup C = \{2,3,4,5,6,7,8\}$.
3. $(A \cap B) \cup (A \cap C) = \{5\} \cup \{5,7\} = \{5,7\}$.
4. $A - C = \{1\}$.
5. $(A \cup C) - (B \cap C) = \{1,3,4,5,6,7\} - \{5\} = \{1,3,4,6,7\}$.
6. $\widetilde{A} = \mathcal{U} - A = \{2,3,4,6,8\}$.
7. $\widetilde{A} \cap \widetilde{B} = \{2,3,4,6,8\} \cap \{1,3,4,6,7\} = \{3,4,6\}$.

Example 2.2.2

Let $A \subseteq B \cup C$ and $A \cap B = \phi$. Show that $A \subseteq C$.

Solution:

Let $x \in A$. Since $A \subseteq B \cup C$, $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$, contradiction. Thus, $x \in C$ and therefore, $A \subseteq C$.

Example 2.2.3

Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Solution:

$$
\begin{aligned}
\text{Let } X \in \mathcal{P}(A \cap B) &\text{ iff } X \subseteq A \cap B \\
&\text{ iff } X \subseteq A \text{ and } X \subseteq B \\
&\text{ iff } X \in \mathcal{P}(A) \text{ and } X \in \mathcal{P}(B) \\
&\text{ iff } X \in \mathcal{P}(A) \cap \mathcal{P}(B).
\end{aligned}
$$

Example 2.2.4

Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Is $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ in general? Explain.

Solution:

$$\begin{aligned}
\text{Let } X \in \mathcal{P}(A) \cup \mathcal{P}(B) \quad &\Rightarrow \quad X \in \mathcal{P}(A) \text{ or } X \in \mathcal{P}(B) \\
&\Rightarrow \quad X \subseteq A \text{ or } X \subseteq B \\
&\Rightarrow \quad X \subseteq A \cup B \\
&\Rightarrow \quad X \in \mathcal{P}(A \cup B).
\end{aligned}$$

In general, $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ and thus $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

For instance, consider $A = \{a\}$ and $B = \{b\}$. Then $A \cup B = \{a, b\}$, $\mathcal{P}(A) = \{\phi, \{a\}\}$ and $\mathcal{P}(B) = \{\phi, \{b\}\}$. Therefore,

$$\mathcal{P}(A \cup B) = \{\phi, \{a\}, \{b\}, \{a, b\}\} \neq \mathcal{P}(A) \cup \mathcal{P}(B) = \{\phi, \{a\}, \{b\}\}.$$

Remark 2.2.1

If $A \subseteq B$, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

Exercise 2.2.1

Suppose that A, B, and C are three nonempty sets. Show that if $A \subseteq B$, then $A - C \subseteq B - C$.

Exercise 2.2.2

Suppose that A, and B are two nonempty sets. Show that $A - B = \phi$ iff $A \cap B = A$.