

Section 2.3: Extended Set Operations

Definition 2.3.1

Let \mathcal{I} be a nonempty set. Suppose that for each $i \in \mathcal{I}$, there is a corresponding set A_i . Then, the family of sets $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ is called an **indexed family of sets**. Each $i \in \mathcal{I}$ is called an **index** and \mathcal{I} is called an **indexing set**. Then

1. The **union over \mathcal{A}** is defined by

$$\bigcup_{i \in \mathcal{I}} A_i = \{x : (\exists A_i \in \mathcal{A}) [x \in A_i]\} = \{x : (\exists A_i) [A_i \in \mathcal{A} \wedge x \in A_i]\}.$$

2. the **intersection over \mathcal{A}** is defined by

$$\bigcap_{i \in \mathcal{I}} A_i = \{x : (\forall A_i \in \mathcal{A}) [x \in A_i]\} = \{x : (\forall A_i) [A_i \in \mathcal{A} \Rightarrow x \in A_i]\}.$$

3. The indexed family \mathcal{A} of sets is said to be **pairwise disjoint** if and only if for all i and j in \mathcal{I} , either $A_i = A_j$ or $A_i \cap A_j = \emptyset$.

Example 2.3.1

Let $\mathcal{I} = \{1, 2, 3\}$, and define $A_i = \{i, i + 1\}$ for each $i \in \mathcal{I}$. Find $\bigcup_{i \in \mathcal{I}} A_i$ and $\bigcap_{i \in \mathcal{I}} A_i$.

Solution:

Note that $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, and $A_3 = \{3, 4\}$. Thus, $\bigcup_{i \in \mathcal{I}} A_i = \{1, 2, 3, 4\}$, and $\bigcap_{i \in \mathcal{I}} A_i = \emptyset$.

Example 2.3.2

For each $i \in \mathbb{N}$, let $A_i = \{j \in \mathbb{N} : j \leq i\}$. Find $\bigcup_{i \in \mathbb{N}} A_i$ and $\bigcap_{i \in \mathbb{N}} A_i$.

Solution:

Note that $A_1 = \{1\}$, $A_2 = \{1, 2\}$, \dots , $A_n = \{1, 2, \dots, n\}$ and so on. Thus, $\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$ while $\bigcap_{i \in \mathbb{N}} A_i = \{1\}$.

Theorem 2.3.1

Let $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ be an indexed family of sets. Then,

Proof:

Proof of (1): Let $x \in A_k$. Since $A_k \in \mathcal{A}$, $x \in \bigcup_{i \in \mathcal{I}} A_i$. Thus, $A_k \subseteq \bigcup_{i \in \mathcal{I}} A_i$.

Proof of (2): Let $x \in \bigcap_{i \in \mathcal{I}} A_i$. Then, $x \in A_i$ for every $i \in \mathcal{I}$. Since $k \in \mathcal{I}$, $x \in A_k$. Thus,

$$\bigcap_{i \in \mathcal{I}} A_i \subseteq A_k.$$

Proof of (3.a):

$$\begin{aligned}
x \in \widetilde{\bigcup_{i \in \mathcal{I}} A_i} &\Leftrightarrow x \notin \bigcup_{i \in \mathcal{I}} A_i \\
&\Leftrightarrow x \notin A_i \text{ for all } i \in \mathcal{I} \\
&\Leftrightarrow x \in \widetilde{A_i} \text{ for all } i \in \mathcal{I} \\
&\Leftrightarrow x \in \bigcap_{i \in \mathcal{I}} \widetilde{A_i}.
\end{aligned}$$

Proof of (3.b): A similar proof as that in part (3.a) can be shown in this part as well. However, we use a different style as follows: Using $A_i = \widetilde{\bar{A}}_i$ together with part (3.a) of this theorem, we get

$$\widetilde{\bigcap_{i \in \mathcal{I}} A_i} = \widetilde{\bigcap_{i \in \mathcal{I}} \widetilde{A}_i} = \widetilde{\bigcup_{i \in \mathcal{I}} \widetilde{A}_i} = \bigcup_{i \in \mathcal{I}} \widetilde{A}_i.$$

Example 2.3.3

Let $\mathcal{I} = \{1, 2, 3, 4\}$ so that $A_1 = \{1, 2, 7\}$, $A_2 = \{3, 4, 8\}$, $A_3 = \{1, 4, 8\}$, and $A_4 = \{1, 3, 4, 7\}$.

If $\mathcal{U} = \{1, 2, 3, \dots, 10\}$, answer each of the following:

$$a. \quad \bigcup_{i \in \mathcal{I}} A_i = \{1, 2, 3, 4, 7, 8\}.$$

- b. $\bigcap_{i \in \mathcal{I}} A_i = \phi.$
- c. $\bigcup_{i \in \mathcal{I}} \widetilde{A}_i = \widetilde{\bigcap_{i \in \mathcal{I}} A_i} = \mathcal{U}.$
- d. $\bigcap_{i \in \mathcal{I}} \widetilde{A}_i = \widetilde{\bigcup_{i \in \mathcal{I}} A_i} = \{5, 6, 9, 10\}.$
- e. Is $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ a pairwise disjoint? Explain. Answer: No, $A_3 \cap A_4 = \{1, 4\} \neq \phi.$

Example 2.3.4

Let $\mathcal{U} = \mathbb{N}$ and $\mathcal{I} = \mathbb{N}$. Define $A_i = \mathbb{N} - \{1, 2, \dots, i\}$ for all $i \in \mathcal{I}$. Find:

- a. $A_{10} = \{11, 12, 13, \dots\}.$
- b. $\bigcup_{i \in \mathcal{I}} A_i = \{2, 3, 4, 5, \dots\}.$
- c. $\bigcap_{i \in \mathcal{I}} A_i = \phi.$

Example 2.3.5

If $\mathcal{U} = \mathbb{R}$, let $A_n = [-\frac{1}{n}, 2 + \frac{1}{n})$ for all $n \in \mathbb{N}$. Find:

- a. $\bigcup_{n \in \mathbb{N}} A_n = [-1, 3) =: A_1.$
- b. $\bigcap_{n \in \mathbb{N}} A_n = [0, 2].$
- c. $\bigcap_{n \in \mathbb{N}} \widetilde{A}_n = \widetilde{\bigcup_{n \in \mathbb{N}} A_n} = \mathbb{R} - [-1, 3).$
- d. $\bigcup_{n \in \mathbb{N}} \widetilde{A}_n = \widetilde{\bigcap_{n \in \mathbb{N}} A_n} = \mathbb{R} - [0, 2].$

Example 2.3.6

Let $\mathcal{U} = \mathbb{R}$ and define $S_a = (-a, a)$ for all $a \in \mathbb{N}$. Find

- a. $\bigcup_{a \in \mathbb{N}} S_a = \mathbb{R}.$

b. $\bigcap_{a \in \mathbb{N}} S_a = (-1, 1)$.

Exercise 2.3.1

Let $\mathcal{A} = \{ A_i : i \in \mathcal{I} \}$ be an indexed family of sets for a nonempty set \mathcal{I} . Show that if $B \subseteq A_i$ for every $i \in \mathcal{I}$, then $B \subseteq \bigcap_{i \in \mathcal{I}} A_i$.

Exercise 2.3.2

For each natural number $n \geq 3$, let $A_n = \left[\frac{1}{n}, 2 + \frac{1}{n} \right]$, and $\mathcal{A} = \{ A_n : n \geq 3 \}$. Find $\bigcap_{n \geq 3} A_n$ and $\bigcup_{n \geq 3} A_n$.