Section 2.4: Proof by Induction

Definition 2.4.1: Principle of Mathematical Induction (PMI)

If S is a subset of $\mathbb{N}$ so that:

1. $1 \in S$, and
2. for all $n \in \mathbb{N}$, if $n \in S$, then $n+1 \in S$,

then $S = \mathbb{N}$.

2.4.1 Proof of $(\forall n \in \mathbb{N})\mathbf{P}(n)$ using PMI

- **Basic Step**: Show that $\mathbf{P}(1)$ is true.
- **Induction Step**: Show that for all $n \in \mathbb{N}$, if $\mathbf{P}(n)$ is true, then $\mathbf{P}(n+1)$ is true.
- **Conclusion**: By step 1 and step 2 and using the PMI, $\mathbf{P}(n)$ is true for all $n \in \mathbb{N}$.

Example 2.4.1

Show that for all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution:

For $n = 1$, clearly $1 = \frac{1(1+1)}{2}$ is true. Assume that for some $n \in \mathbb{N}$, we have

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Now, we want to show that $1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$.

$$\begin{aligned}
\overbrace{1 + 2 + 3 + \cdots + n}^{\text{use our assumption}} + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}.
\end{aligned}$$

Example 2.4.2

Show that for all $n \in \mathbb{N}$, $\sum_{i=1}^{n}(2i-1) = n^2$.

Solution:

For $n = 1$, $2(1) - 1 = 1 = 1^2$, which is true. Assume that for some $n \in \mathbb{N}$, we have $\sum_{i=1}^{n}(2i-1) = n^2$. We want to show that $\sum_{i=1}^{n+1}(2i-1) = (n+1)^2$. Thus,

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2.$$

Example 2.4.3

Show that for all $n \in \mathbb{N}$, $n + 3 < 5n^2$.

Solution:

For $n = 1$ we have $1 + 3 = 4 < 5$ which is true. So, assume that for n, $n + 3 < 5n^2$ is true. For $n + 1$, we want to show that $(n + 1) + 3 < 5(n + 1)^2 = 5n^2 + 10n + 5$. Then,

$$(n+1) + 3 = (n+3) + 1 < 5n^2 + 1 < 5n^2 + (10n + 4) + 1 = 5(n+1)^2.$$

Therefore, for all $n \in \mathbb{N}$, $n + 3 < 5n^2$.

Definition 2.4.2

For $n \in \mathbb{N}$, define $0! = 1$ and $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots 2 \cdot 1$. Then, the **bionomial coefficient "n choose k"**, where $0 \leq k \leq n$, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)(n-3)\cdots(n-k+2)(n-k+1)}{k!}.$$

Moreover, the **bionomial expansion** of any $a, b \in \mathbb{R}$ is given by

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \, b^{n-k}.$$

Remark 2.4.1: Pascal's Triangle

Let $a, b \in \mathbb{R}$. Then, the coefficients of the bionomial expansion $(a+b)^n$ can be computed by the Pascal's Triangle for each n.

$$\begin{array}{cccccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Example 2.4.4

Show that for all $n \in \mathbb{N}$, $\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15}$ is an integer.

Solution:

$\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15} = \dfrac{5n^3 + 3n^5 + 7n}{15}$ is an integer iff $15 \mid 5n^3 + 3n^5 + 7n$ iff $\exists k \in \mathbb{N}$ such that $5n^3 + 3n^5 + 7n = 15k$.

For $n = 1$, we have $5 + 3 + 7 = 15$ which is true. So assume that there $k \in \mathbb{N}$ such that $5n^3 + 3n^5 + 7n = 15k$. Then, we want to show that

$$5(n+1)^3 + 3(n+1)^5 + 7(n+1) = 15h \tag{2.4.1}$$

for some $h \in \mathbb{N}$. Thus, using the Pascal's Triangle we get

$$\begin{aligned}
\text{Eqn.}(2.4.1) &= 5(n^3 + 3n^2 + 3n + 1) + 3(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) + 7n + 7 \\
&= \underbrace{(5n^3 + 3n^5 + 7n)}_{=15k} + 15n^2 + 15n + 5 + 15n^4 \\
&\quad + 30n^3 + 30n^2 + 15n + 3 + 7 \\
&= 15k + 15\left[n^2 + n + n^4 + 2n^3 + 2n^2 + n + 1\right]
\end{aligned}$$

Thus $15 \mid 5(n+1)^3 + 3(n+1)^5 + 7(n+1)$ and $\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15}$ is an integer for all $n \in \mathbb{N}$.

Example 2.4.5

Express the terms of $(2x - 4yz^2)^5$ for $x, y, z \in \mathbb{R}$.

Solution:

Let $a = 2x$, $b = -4yz^2$, and $n = 5$. Using the bionomial expansion form, we get

$$\begin{aligned}(2x - 4yz^2)^5 &= (2x)^5 + 5(2x)^4(-4yz^2) + 10(2x)^3(-4yz^2)^2 + 10(2x)^2(-4yz^2)^3 \\ &\quad +5(2x)(-4yz^2)^4 + (-4yz^2)^5.\end{aligned}$$

Definition 2.4.3: Generalized Principle of Mathematical Induction (GPMI)

Let k be a natural number. If S is a subset of $\mathbb{N}$ so that:

1. $k \in S$, and
2. for all $n \in \mathbb{N}$ with $n \geq k$, if $n \in S$, then $n + 1 \in S$,

then S contains all natural number greater than or equal to k.

Example 2.4.6

Show that for all $n \geq 5$, $n^2 - n - 20 \geq 0$.

Solution:

For $n = 5$, we have $25 - 5 - 20 = 0 \geq 0$ which is true. Assume that for some $n \geq 5$, $n^2 - n - 20 \geq 0$ is true. For $n + 1$, we have

$$(n+1)^2 - (n+1) - 20 = n^2 + 2n + \cancel{1} - n - \cancel{1} - 20 = (n^2 - n - 20) + \underbrace{2n}_{\text{positive}} \geq 0.$$

Thus, $n^2 - n - 20 \geq 0$ for all $n \geq 5$.

Example 2.4.7

Let $n \in \mathbb{N}$. Show that $(n + 1)! > 2^{n+3}$ for all $n \geq 5$.

Solution:

For $n = 5$, we have $6! = 720 \geq 2^8 = 256$ which is true. Assume that for some $n \geq 5$, $(n + 1)! > 2^{n+3}$ is true.

For $n+1$, we want to show that $(n+2)! > 2^{n+4}$ for all $n+1 \geq 5$. Since $n+2 > 2$ for all $n \geq 4$, we get

$$(n+2)! = (n+2)(n+1)! > (n+2)2^{n+3} > 2 \cdot 2^{n+3} = 2^{n+4}.$$

Thus, $(n+1)! > 2^{n+3}$ for all $n \geq 5$.