Chapter 3

Chapter

3 Relations

Section 3.1: Cartesian Products and Relations

Definition 3.1.1

Let A and B be two sets. An **ordered pair** is $(a, b) \neq \{a, b\}$ for $a \in A$ and $b \in B$. We say that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Definition 3.1.2

Let A and B be two sets. The (**Cartesian** or **cross**) **product** of A and B, denoted by $A \times B$, is defined by

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Moreover, if $(a, b) \in A \times B$, then $a \in A$ and $b \in B$. If $(a, b) \notin A \times B$, then either $a \notin A$ or $b \notin B$.

Remark 3.1.1

Let A and B be two given sets. Then,

1. if A has m elements and B has n elements, then $A \times B$ has mn elements.
2. In general, $A \times B \neq B \times A$.

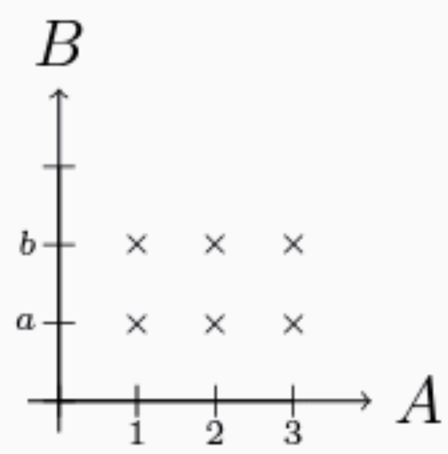
Example 3.1.1

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find $A \times B$ and $B \times A$.

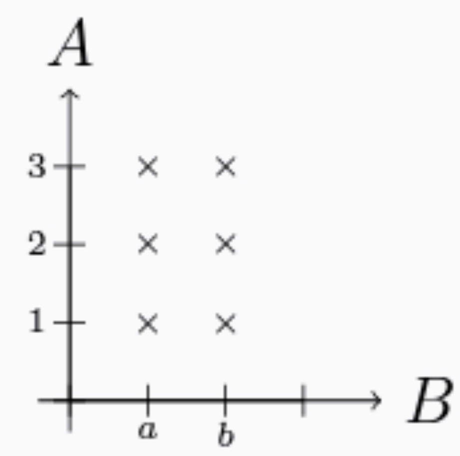
Solution:

Note that, in general $A \times B \neq B \times A$ as this example shows.

$$\begin{aligned} A \times B &= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}, \text{ and} \\ B \times A &= \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}. \end{aligned}$$



$A \times B$



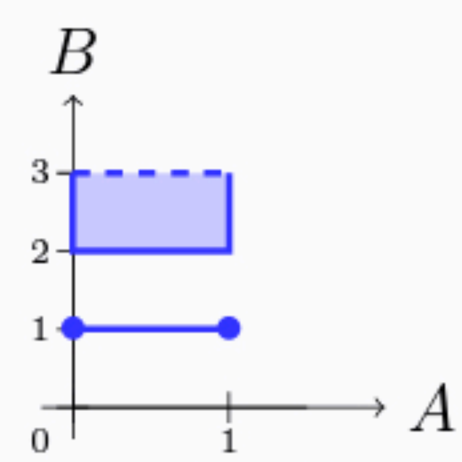
$B \times A$

Example 3.1.2

Let $A = [0, 1]$ and $B = \{1\} \cup [2, 3)$. Find $A \times B$.

Solution:

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.



$A \times B$

Theorem 3.1.1

If A and B are nonempty set, then $A \times B = B \times A$ iff $A = B$.

Proof:

„ $\Rightarrow$ ": Assume that $A \neq \phi$, $B \neq \phi$ and $A \times B = B \times A$. Let $a \in A$, then there is $b \in B$ such that $(a, b) \in A \times B = B \times A$ which implies that $a \in B$ Thus, $A \subseteq B$.

Let $b \in B$, then there is $a \in A$ such that $(b, a) \in B \times A = A \times B$ which implies that $b \in A$. Thus, $B \subseteq A$ and therefore $A = B$.

„ $\Leftarrow$ ": if $A = B$, then $A \times B = A \times A = B \times A$.

Theorem 3.1.2

Let A, B, C, and D be sets. Then

$$1. \begin{cases} \text{a. } A \times (B \cup C) & = & (A \times B) \cup (A \times C). \\ \text{b. } (A \cup B) \times C & = & (A \times C) \cup (B \times C). \\ \text{c. } A \times (B \cap C) & = & (A \times B) \cap (A \times C). \\ \text{d. } (A \cap B) \times C & = & (A \times C) \cap (B \times C). \end{cases}$$

2. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

3. $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof:

Proof of (1.a):

$$\begin{aligned}
(x,y) \in A \times (B \cup C) \quad &\text{iff} \quad x \in A \wedge y \in B \cup C \\
&\text{iff} \quad x \in A \wedge (y \in B \vee y \in C) \\
&\text{iff} \quad (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
&\text{iff} \quad ((x,y) \in A \times B) \vee ((x,y) \in A \times C) \\
&\text{iff} \quad (x,y) \in (A \times B) \vee (A \times C).
\end{aligned}$$

Proof of (2):

$$\begin{aligned}
(x,y) \in (A \times B) \cap (C \times D) \quad &\text{iff} \quad (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D) \\
&\text{iff} \quad (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D) \\
&\text{iff} \quad (x \in A \cap C) \wedge (y \in B \cap D) \\
&\text{iff} \quad (x,y) \in (A \cap C) \times (B \cap D).
\end{aligned}$$

Proof of (3): Let $(x,y) \in (A \times B) \cup (C \times D)$, then $(x,y) \in A \times B$ or $(x,y) \in C \times D$.

Case(i): $(x,y) \in A \times B$ implies that $x \in A$ and $y \in B$. Then, $x \in A \cup C$ and $y \in B \cup D$. Thus, $(x,y) \in (A \cup C) \times (B \cup D)$.

Case(ii): $(x,y) \in C \times D$ implies that $x \in C$ and $y \in D$. Then again $x \in A \cup C$ and $y \in B \cup D$. Thus, $(x,y) \in (A \cup C) \times (B \cup D)$.

Therefore, $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Remark 3.1.2

Note that $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$: For instance, Let $A = B = \{0\}$, and $C = D = \{1\}$. Then, $(0,1) \in (A \cup C) \times (B \cup D)$ while $(0,1) \notin (A \times B) \cup (C \times D)$. Therefore, $(A \cup C) \times (B \cup D) \not\subseteq (A \times B) \cup (C \times D)$.

Definition 3.1.3

Let A and B be sets. A **relation** $\mathcal{R}$ from A to B is a subset of $A \times B$. In this case, we write $a\mathcal{R}b$ for $(a,b) \in \mathcal{R}$ and say that "a is related to b". Also, $a\not\mathcal{R}b$ means that $(a,b) \notin \mathcal{R} \subseteq A \times B$. Moreover, if $A = B$, then subsets of $A \times A$ are called relations on A.

Definition 3.1.4

If $\mathcal{R} \subseteq A \times B$ is a relation, then the **domain** of $\mathcal{R}$ is $\mathrm{Dom}(\mathcal{R}) = \{a \in A : (a,b) \in \mathcal{R}\}$. Moreover, the **range** of $\mathcal{R}$ is $\mathrm{Rng}(\mathcal{R}) = \{b \in B : (a,b) \in \mathcal{R}\}$.

Example 3.1.3

Let $A = \{1, 2, \{3\}, 4\}$ and $B = \{a, b, c, d\}$. Find the domain and range of $\mathcal{R}$, where

$$\mathcal{R} = \{(1,c), (\{3\},a), (1,d), (2,d)\} \subseteq A \times B.$$

Solution:

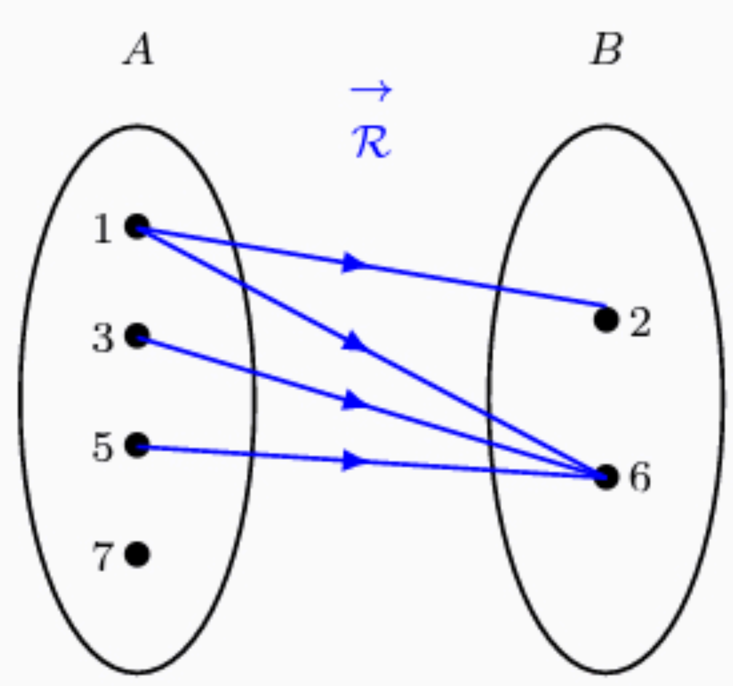
The $\mathrm{Dom}(\mathcal{R}) = \{1, 2, \{3\}\} \subseteq A$ and the $\mathrm{Rng}(\mathcal{R}) = \{a, c, d\} \subseteq B$. Note that $\mathrm{Dom}(\mathcal{R}) \neq A$ and $\mathrm{Rng}(\mathcal{R}) \neq B$.

Example 3.1.4

Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 6\}$. Let $\mathcal{R} \subseteq A \times B$ defined by $\mathcal{R} = \{(a,b) \in A \times B : a < b\}$. Find $\mathcal{R}$ along with its domain and range.

Solution:

$$\mathcal{R} = \{(1,2), (1,6), (3,6), (5,6)\}$$
$$\mathrm{Dom}(\mathcal{R}) = \{1, 3, 5\}$$
$$\mathrm{Rng}(\mathcal{R}) = \{2, 6\}.$$



Example 3.1.5

Let $\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2 + 3\}$. Find the domain and the range of the relation $\mathcal{R}$.

Solution:

Domain: $x \in \text{Dom}(\mathcal{R})$ iff $\exists y \in \mathbb{R}$ with $y = x^2 + 3$ which is true for all $x \in \mathbb{R}$. Thus, $\text{Dom}(\mathcal{R}) = \mathbb{R}$. Range: $y \in \text{Rng}(\mathcal{R})$ iff $\exists x \in \mathbb{R}$ with $y = x^2 + 3$ and since $x^2 \geq 0$, we have $y \geq 3$. Therefore, $\text{Rng}(\mathcal{R}) = [3, \infty)$.

Definition 3.1.5

For any set A, the relation $\mathcal{I}_A$ is the **identity relation** on A and is defined by

$$\mathcal{I}_A = \{(a, a) : a \in A\},$$

with $\text{Dom}(\mathcal{I}_A) = A = \text{Rng}(\mathcal{I}_A)$.

Definition 3.1.6

For any sets A and B, if $\mathcal{R} \subseteq A \times B$ is a relation, then the **inverse relation** is

$$\mathcal{R}^{-1} = \{(b, a) : (a, b) \in \mathcal{R}\} \subseteq B \times A,$$

with $\text{Dom}(\mathcal{R}^{-1}) = \text{Rng}(\mathcal{R})$ and $\text{Rng}(\mathcal{R}^{-1}) = \text{Dom}(\mathcal{R})$.

Definition 3.1.7

Let $\mathcal{R} \subseteq A \times B$ be a relation and let $\mathcal{S} \subseteq B \times C$ be a relation. The **composition relation** $\mathcal{S} \circ \mathcal{R}$ is defined by

$$\mathcal{S} \circ \mathcal{R} = \{(a, c) : (\exists b \in B)\big((a, b) \in \mathcal{R} \text{ and } (b, c) \in \mathcal{S}\big)\} \subseteq A \times C.$$

Moreover, $\text{Dom}(\mathcal{S} \circ \mathcal{R}) \subseteq \text{Dom}(\mathcal{R})$.

Example 3.1.6

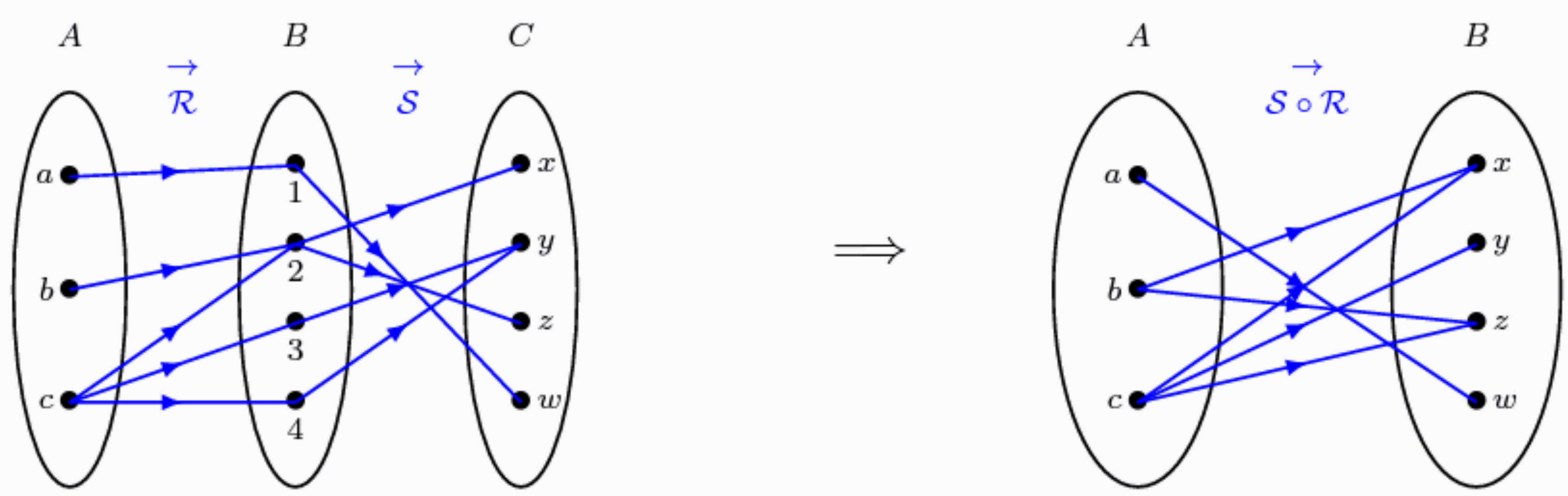
Let $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $C = \{x, y, z, w\}$. Let

$$\mathcal{R} = \{(a,1),(b,2),(c,2),(c,3),(c,4)\} \subseteq A \times B, \text{ and}$$
$$\mathcal{S} = \{(1,w),(2,x),(2,z),(3,y),(4,y)\} \subseteq B \times C.$$

Find $\mathcal{R}^{-1}$, and $\mathcal{S} \circ \mathcal{R}$.

Solution:

$$\begin{aligned}\mathcal{R}^{-1} &= \{(1,a),(2,b),(2,c),(3,c),(4,c)\} \subseteq B \times A.\\ \mathcal{S} \circ \mathcal{R} &= \{(a,w),(b,x),(b,z),(c,x),(c,z),(c,y)\} \subseteq A \times C.\end{aligned}$$



Example 3.1.7

Let $\mathcal{R} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x < y\}$. Find $\mathcal{R}^{-1}$.

Solution:

Note that

$$\begin{aligned}(x,y) \in \mathcal{R}^{-1} \quad &\text{iff} \quad (y,x) \in \mathcal{R}\\ &\text{iff} \quad y < x\\ &\text{iff} \quad x > y.\end{aligned}$$

That is $\mathcal{R}^{-1} = \{\, (x,y) \in \mathbb{R} \times \mathbb{R} : x > y \,\}$.