Example 3.1.8

Let $\mathcal{R} = \{\,(x,y) \in \mathbb{R}\times\mathbb{R} : y = x-1\,\}$ and let $\mathcal{S} = \{\,(x,y) \in \mathbb{R}\times\mathbb{R} : y = x^2\,\}$. Find $\mathcal{S}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{S}$.

Solution:

$$\begin{aligned}
\mathcal{S}\circ\mathcal{R} &= \{\,(x,y) : (\,\exists z\in\mathbb{R}\,)(\,(x,z)\in\mathcal{R} \text{ and } (z,y)\in\mathcal{S}\,)\,\}\\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,z = x-1 \text{ and } y = z^2\,\Big)\,\Big\}\\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,y = (x-1)^2\,\Big)\,\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}\circ\mathcal{S} &= \{\,(x,y) : (\,\exists z\in\mathbb{R}\,)(\,(x,z)\in\mathcal{S} \text{ and } (z,y)\in\mathcal{R}\,)\,\}\\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,z = x^2 \text{ and } y = z-1\,\Big)\,\Big\}\\
&= \Big\{\,(x,y) : (\,\exists z\in\mathbb{R}\,)\Big(\,y = x^1-1\,\Big)\,\Big\}
\end{aligned}$$

Theorem 3.1.3

Let A, B, C, and D be sets. Let $\mathcal{R}\subseteq A\times B$, $\mathcal{S}\subseteq B\times C$, and $\mathcal{T}\subseteq C\times D$. Then,

1. $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$.
2. $\mathcal{T}\circ(\mathcal{S}\circ\mathcal{R}) = (\mathcal{T}\circ\mathcal{S})\circ\mathcal{R}$.
3. $(\mathcal{S}\circ\mathcal{R})^{-1} = \mathcal{R}^{-1}\circ\mathcal{S}^{-1}$.

Proof:

Proof of part(2): Let $a\in A$ and $d\in D$ so that

$$\begin{aligned}
(a,d)\in\mathcal{T}\circ(\mathcal{S}\circ\mathcal{R}) \quad &\text{iff} \quad (\exists c\in C)\Big[(a,c)\in\mathcal{S}\circ\mathcal{R} \text{ and } (c,d)\in\mathcal{T}\Big]\\
&\text{iff} \quad (\exists c\in C)\Big[(\exists b\in B)\Big((a,b)\in\mathcal{R} \text{ and } (b,c)\in\mathcal{S}\Big) \text{ and } (c,d)\in\mathcal{T}\Big]\\
&\text{iff} \quad (\exists c\in C)(\exists b\in B)\Big[(a,b)\in\mathcal{R} \text{ and } (b,c)\in\mathcal{S} \text{ and } (c,d)\in\mathcal{T}\Big]\\
&\text{iff} \quad (\exists b\in B)\Big[(a,b)\in\mathcal{R} \text{ and } (\exists c\in C)\Big((b,c)\in\mathcal{S} \text{ and } (c,d)\in\mathcal{T}\Big)\Big]\\
&\text{iff} \quad (\exists b\in B)\Big[(a,b)\in\mathcal{R} \text{ and } (b,d)\in\mathcal{T}\circ\mathcal{S}\Big]\\
&\text{iff} \quad (a,d)\in(\mathcal{T}\circ\mathcal{S})\circ\mathcal{R}.
\end{aligned}$$

Proof of part (3): Let $a \in A$ and $c \in C$ so that

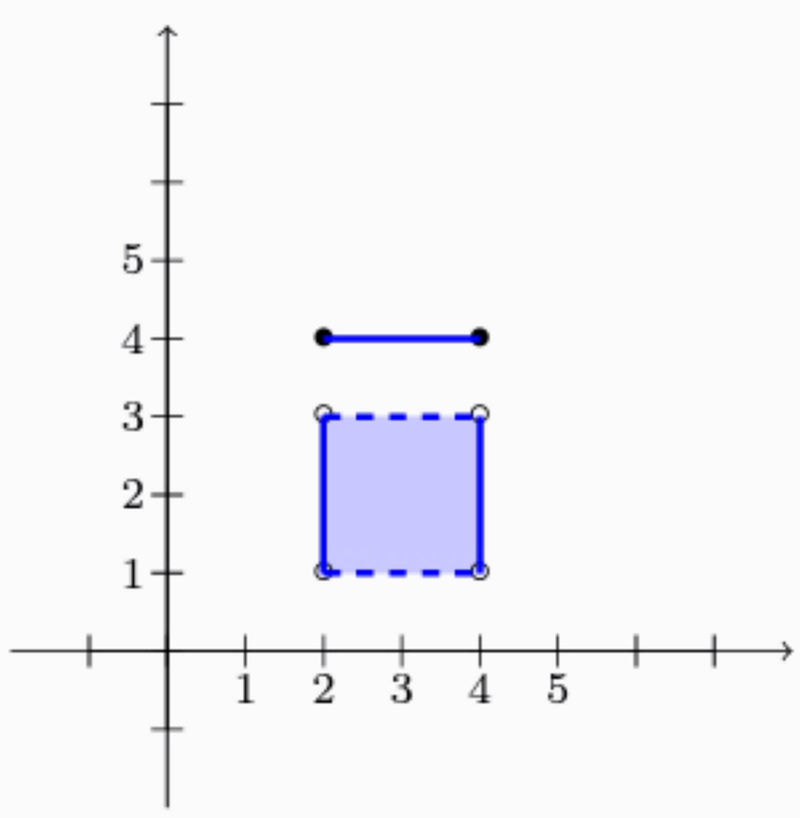
$$\begin{aligned}(c,a) \in (\mathcal{S} \circ \mathcal{R})^{-1} \quad &\text{iff} \quad (a,c) \in \mathcal{S} \circ \mathcal{R}\\ &\text{iff} \quad (\exists b \in B)\Big[(a,b) \in \mathcal{R} \text{ and } (b,c) \in \mathcal{S}\Big]\\ &\text{iff} \quad (\exists b \in B)\Big[(b,a) \in \mathcal{R}^{-1} \text{ and } (c,b) \in \mathcal{S}^{-1}\Big]\\ &\text{iff} \quad (\exists b \in B)\Big[(c,b) \in \mathcal{S}^{-1} \text{ and } (b,a) \in \mathcal{R}^{-1}\Big]\\ &\text{iff} \quad (c,a) \in \mathcal{R}^{-1} \circ \mathcal{S}^{-1}.\end{aligned}$$

Example 3.1.9

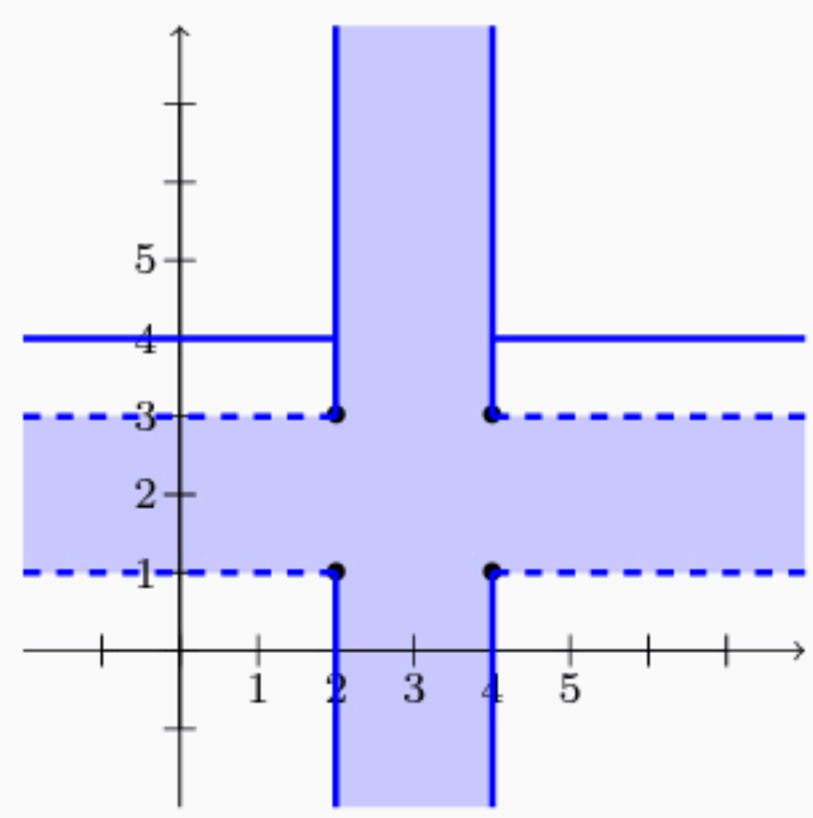
Let $A = [2,4]$ and $B = (1,3) \cup \{4\}$. Let $\mathcal{R}$ be the relation on $A \times \mathbb{R}$ with $x\mathcal{R}y$ iff $x \in A$ and let $\mathcal{S}$ be the relation on $\mathbb{R} \times B$ with $x\mathcal{S}y$ iff $y \in B$. Find $\mathcal{R} \cap \mathcal{S}$ and $\mathcal{R} \cup \mathcal{S}$.

Solution:

By Theorem 3.1.2 part(2), $\mathcal{R} \cap \mathcal{S} = (A \times \mathbb{R}) \cap (\mathbb{R} \times B) = (A \cap \mathbb{R}) \times (\mathbb{R} \cap B) = A \times B$. Therefore, $\mathcal{R} \cap \mathcal{S} = A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$. On the other hand, $\mathcal{R} \cup \mathcal{S} = \{(a,b) \in \mathbb{R} \times \mathbb{R} : a \in A \text{ or } b \in B\}$.



$\mathcal{R} \cap \mathcal{S}$



$\mathcal{R} \cup \mathcal{S}$

Exercise 3.1.1

Let A and B be two nonempty sets. Show that if $A \times B \subseteq B \times C$, then $A \subseteq C$.

Exercise 3.1.2

Let $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times C$ be two relations. Show that $\mathrm{Dom}(\mathcal{S} \circ \mathcal{R}) \subseteq \mathrm{Dom}(\mathcal{R})$.

Section 3.2: Equivalence Relations

Definition 3.2.1

Let A be a set and $\mathcal{R}$ be a relation on A. Then $\mathcal{R}$ is called an **equivalence relation** if and only if:

1. $\mathcal{R}$ is **reflexive** on A: $(\forall x \in A)\ x\mathcal{R}x$.
2. $\mathcal{R}$ is **symmetric** on A: $(\forall x, y \in A)$ if $x\mathcal{R}y$, then $y\mathcal{R}x$.
3. $\mathcal{R}$ is **transitive** on A: $(\forall x, y, z \in A)$ if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

Example 3.2.1

Let $A = \{1,2,3,4\}$ and $\mathcal{R}_1 = \{(1,2),(2,3),(1,3)\}$, $\mathcal{R}_2 = \{(1,1),(1,2)\}$, $\mathcal{R}_3 = \{(3,4)\}$, $\mathcal{R}_4 = \{(1,2),(2,1)\}$, and $\mathcal{R}_5 = \{(1,1),(2,2),(3,3),(4,4)\}$. Decide which relation is reflexive, symmetric, transitive.

Solution:

$\mathcal{R}_5$ is reflexive. $\mathcal{R}_4$, and $\mathcal{R}_5$ are symmetric. $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_5$ are transitive. Therefore, $\mathcal{R}_5$ is an equivalence relation on A.

Example 3.2.2

Let $\mathcal{R} = \{(x,y) : xy > 0\}$ be a relation on $\mathbb{Z}$. Discuss whether $\mathcal{R}$ reflexive, symmetric, transitive, and equivalence relation.

Solution:

Clearly, $x\mathcal{R}x$ for all $x \in \mathbb{Z}$ except for $x = 0$, thus $\mathcal{R}$ is not reflexive. If $x\mathcal{R}y$, then $xy > 0$ or $yx > 0$ which imples that $y\mathcal{R}x$. Thus, $\mathcal{R}$ is symmetric. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $xy > 0$ and $yz > 0$. Considering the cases of $y \in \mathbb{Z} - \{0\}$, we have

1. case 1: $y > 0$, then $x > 0$ and $z > 0$ which implies that $xz > 0$ and thus $x\mathcal{R}z$.
2. case 1: $y < 0$, then $x < 0$ and $z < 0$ which implies that $xz > 0$ and thus $x\mathcal{R}z$.

In either cases, $\mathcal{R}$ is transitive on $\mathbb{Z}$. Note that $\mathcal{R}$ is not reflexive and thus it is not an equivalence relation on $\mathbb{Z}$.

Example 3.2.3

Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ given by $x\mathcal{R}y$ iff $x - y$ is even. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

Solution:

Reflexive: Since $x - x = 0$ is even, $x\mathcal{R}x$ for all $x \in \mathbb{Z}$. Thus, $\mathcal{R}$ is reflexive.

Symmetric: Assume that $x\mathcal{R}y$, then there is $k \in \mathbb{Z}$ such that $x - y = 2k$. Thus, $y - x = 2(-k)$ which implies that $y\mathcal{R}x$. Thus, $\mathcal{R}$ is symmetric.

Transitive: Let $x\mathcal{R}y$ and $y\mathcal{R}z$. Then, there are $h, k \in \mathbb{Z}$ such that $x - y = 2h$ and $y - z = 2k$. Adding these two equations, we get $x - z = 2(h + k)$ which is even. Therefore, $x\mathcal{R}z$ and $\mathcal{R}$ is transitive.

Therefore, $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

Definition 3.2.2

Let $\mathcal{R}$ be an equivalence relation on a set A. For $x \in A$, define the **equivalence class** of x determined by $\mathcal{R}$ as

$$x/\mathcal{R} = \{y \in A : x\mathcal{R}y\},$$

which reads "the class of x modulo $\mathcal{R}$" or "x mod $\mathcal{R}$. The set of all equivalence classes is called A modulo $\mathcal{R}$ and is defined by

$$A/\mathcal{R} = \{x/\mathcal{R} : x \in A\}.$$

Example 3.2.4

Let $\mathcal{R} = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ be an equivalence relation on $A = \{1,2,3\}$. Find:

- $1/\mathcal{R} = \{1,2\}$.
- $2/\mathcal{R} = \{1,2\}$.
- $3/\mathcal{R} = \{3\}$.
- $A/\mathcal{R} = \{\{1,2\},\{3\}\}$.

Example 3.2.5

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y \Leftrightarrow 2 \mid x+y$. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$. Calculate all the equivalence classes of $\mathcal{R}$.

Solution:

reflexive: Since $x+x=2x$, $2 \mid x+x$ and thus $x\mathcal{R}x$. So, $\mathcal{R}$ is reflexive.

symmetric: if $x\mathcal{R}y$, then $2 \mid x+y$. Thus, $2 \mid y+x$ as well and $y\mathcal{R}x$. Therefore, $\mathcal{R}$ is symmetric.

transitive: Assume that $x\mathcal{R}y$ and $y\mathcal{R}z$. Then $2 \mid x+y$ and $2 \mid y+z$. Thus, $2 \mid x+z+2y$. But because $2 \mid 2y$, we have $2 \mid x+z$. Thus, $x\mathcal{R}z$ and $\mathcal{R}$ is transitive.

Therefore, $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$.

For $x \in \mathbb{N}$, $x/\mathcal{R} = \{y \in \mathbb{N} : 2 \mid x+y\}$. Thus,

$$\overline{1} = \{1,3,5,7,9,\cdots\} = \overline{3} = \overline{5} = \cdots, \text{ and } \overline{2} = \{2,4,6,8,10,\cdots\} = \overline{2} = \overline{4} = \cdots.$$

Therefore, $\mathbb{N} = \overline{1} \cup \overline{2}$.

Theorem 3.2.1

Let $\mathcal{R}$ be an equivalence relation on a nonempty set A. For all $x, y \in A$,

1. $x/\mathcal{R} \subseteq A$ and $x \in x/\mathcal{R} \neq \phi$.
2. $x\,\mathcal{R}\,y$ iff. $x/\mathcal{R} = y/\mathcal{R}$.
3. $x \not\mathcal{R} y$ iff. $x/\mathcal{R} \cap y/\mathcal{R} = \phi$.

Proof:

1. Clearly, $x/\mathcal{R} \subseteq A$ by the definition. Since$\mathcal{R}$ is reflexive, $x\,\mathcal{R}\,x$ and hence $x \in x/\mathcal{R}$.
2. " $\Rightarrow$ ": Suppose $x\,\mathcal{R}\,y$. Then $y\,\mathcal{R}\,x$ (since $\mathcal{R}$ is symmetric). To show that $x/\mathcal{R} = y/\mathcal{R}$, we first show that $x/\mathcal{R} \subseteq y/\mathcal{R}$: Let $z \in x/\mathcal{R} \Rightarrow x\,\mathcal{R}\,z$ and $y\,\mathcal{R}\,x$. Hence, $y\,\mathcal{R}\,z$. Hence, $x/\mathcal{R} \subseteq y/\mathcal{R}$. The proof of $y/\mathcal{R} \subseteq x/\mathcal{R}$ is similar.

   " $\Leftarrow$ ": Suppose $x/\mathcal{R} = y/\mathcal{R}$. Then $x \in x/\mathcal{R} = y/\mathcal{R}$. That is $x\,\mathcal{R}\,y$.
3. " $\Rightarrow$ ": Suppose $x \not\mathcal{R} y$. We proof by contradiction: Assume that there is $z \in x/\mathcal{R} \cap y/\mathcal{R}$. Then, $z \in x/\mathcal{R}$ and $z \in y/\mathcal{R}$ and hence $x\,\mathcal{R}\,z$ and $z\,\mathcal{R}\,y$. Thus, $x\,\mathcal{R}\,y$, contradiction.

   " $\Leftarrow$ ": Suppose $x/\mathcal{R} \cap y/\mathcal{R} = \phi$. Then, $x \in x/\mathcal{R}$. Thus, $x \notin y/\mathcal{R}$ and hence $x \not\mathcal{R} y$.