Definition 3.2.3

Let $m \neq 0$ be a fixed integer. Then "$\equiv_m$" denotes the relation on $\mathbb{Z}$ and is defined by

$$\left(x \equiv y \mod m \text{ or } x \equiv_m y\right) \Leftrightarrow m \mid x - y,$$

which reads "x is congruent to y modulo m". That is $\overline{x} = \{y \in \mathbb{Z} : x \equiv_m y \Leftrightarrow m \mid x - y\}$, and the set of equivalence classes for $\equiv_m$ is $\mathbb{Z}$ mod m (denoted $\mathbb{Z}_m$) and is defined by

$$\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{m-1}\}.$$

Example 3.2.6

Find all the equivalence classes of $\mathbb{Z}_3$.

Solution:

Note that $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$, where $\overline{x} = \{y \in \mathbb{Z} : x \equiv y \mod 3 \text{ or } 3 \mid x - y\}$. Therefore,

- $\overline{0} = 0/\equiv_3 = \{\cdots, -9, -6, -3, 0, 3, 6, 9, \cdots\}$,
- $\overline{1} = 1/\equiv_3 = \{\cdots, -8, -5, -2, 1, 4, 7, 10, \cdots\}$,
- $\overline{2} = 2/\equiv_3 = \{\cdots, -7, -4, -1, 2, 5, 8, 11, \cdots\}$,

Therefore, $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$.

Theorem 3.2.2

Let $m \neq 0$ be a fixed integer. The relation $\equiv_m$ is an equivalence relation on $\mathbb{Z}$. Moreover, $\mathbb{Z}_m$ has m distinct elements: $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \cdots, \overline{m-1}\}$.

Proof:

We only show that $\equiv_m$ is an equivalence relation. reflexive: Since $x - x = 0$ which is divisible by m, $x \equiv_m x$. Thus $\equiv_m$ is reflexive.

symmetric: Assume that $x \equiv_m y$, then $m \mid x - y$ which implies that $m \mid y - x$. Thus, $y \equiv_m x$ and $\equiv_m$ is symmetric.

transitive: Assume that $x \equiv_m y$ and $y \equiv_m z$, then $m \mid x - y$ and $m \mid y - z$. Thus, $m \mid (x - y) + (y - z)$ which implies $m \mid x - z$. Therefore, $x \equiv_m z$ and $\equiv_m$ is transitive. That shows that $\equiv_m$ is an equivalence relation on $\mathbb{Z}$.

Exercise 3.2.1

Let $m \neq 0$. For $x, y \in \mathbb{Z}$: Show that $x \equiv_m y$ if and only if $\overline{x} = \overline{y}$.

Exercise 3.2.2

Let $\mathcal{R}$ be a relation on the set A. Prove that $\mathcal{R} \cup \mathcal{R}^{-1}$ is symmetric.

Exercise 3.2.3

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y$ iff $3 \mid x + y$. Determine whether $\mathcal{R}$ an equivalence relation. Explain.

Exercise 3.2.4

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y$ iff $3 \mid x + 2y$. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{N}$. Find the equivalence class of 1.

Exercise 3.2.5

Let $\mathcal{R}$ be a relation on $\mathbb{R}$ so that $x \, \mathcal{R} \, y$ iff $x = y$ or $xy = 1$. Show that $\mathcal{R}$ is an equivalence relation on $\mathcal{R}$. Find the equivalence classes for 2; 0; and $-\frac{1}{5}$.

Section 3.3: Partitions

Definition 3.3.1

Let A be a set and $\mathcal{A}$ be a family of subsets of A. $\mathcal{A}$ is called a **partition** of A if and only if:

1. if $X \in \mathcal{A}$, then $X \neq \phi$.
2. if $X, Y \in \mathcal{A}$, then either $X = Y$ or $X \cap Y = \phi$.
3. $\bigcup\limits_{X \in \mathcal{A}} X = A$.

Example 3.3.1

1. The set of even natural numbers and odd natural numbers is a partition of $\mathbb{N}$.
2. Let $A_0 = \{0\}$ and $A_i = \{-i, i\}$ for all $i \in \mathbb{N}$. Then $\mathcal{A} = \{A_0, A_1, A_2, A_3, \cdots\}$ is a partition of $\mathbb{Z}$.
3. The set $\{0/\equiv_3, 1/\equiv_3, 2/\equiv_3\}$ is a partition of $\mathbb{Z}$.
4. The set $\{\{$ male students, female students $\}\}$ is a partition for the set of all students in Kuwait University.
5. The collection $\{\, B_i : i \in \mathbb{Z} \,\}$, where $B_i = [i, i+1)$ is a partition of $\mathbb{R}$.

Theorem 3.3.1

Let $A \neq \phi$ and let $\mathcal{R}$ be an equivalence relation on A. Then, the family $A/\mathcal{R} = \{x/\mathcal{R} : x \in A\}$ is a partition of A.

Proof:

Do it your self!

Section 3.4: Ordering Relations

Definition 3.4.1

A relation $\mathcal{R}$ on a set A is called **antisymmetric** if for all $x, y \in A$, if $x\mathcal{R}y$ and $y\mathcal{R}x$, then $x = y$.

Definition 3.4.2

A relation $\mathcal{R}$ on a set A is called a **partial order** (or **partial ordering**) for A if $\mathcal{R}$ is reflexive, antisymmetric, and transitive. In that case, A is called a **partially ordered set** or a **poset**.

Example 3.4.1

Show that "$\subseteq$" is a partial order relation on $\mathcal{P}(A)$ for any set A.

Solution:

reflexive: if $X \in \mathcal{P}(A)$, then $X \subseteq A$ and hence $X \subseteq X$ and hence $x\mathcal{R}x$.

antisymmetric: Let $X, Y \in \mathcal{P}(A)$ with $X\mathcal{R}Y$ and $Y\mathcal{R}X$. Then, $X \subseteq Y$ and $Y \subseteq X$. Therefore, $X = Y$ and $\mathcal{R}$ is antisymmetric.

transitive: Assume that $X, Y, Z \in \mathcal{P}(A)$ with $X \subseteq Y$ and $Y \subseteq Z$. Then $X \subseteq Z$ and hence $X\mathcal{R}Z$.

Therefore, $\mathcal{R}$ is a partial order relation on $\mathcal{P}(A)$.

Example 3.4.2

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $a\mathcal{R}b \Leftrightarrow a \mid b$ for all $a, b \in \mathbb{N}$. Show that $\mathcal{R}$ is a partial order on $\mathbb{N}$.

Solution:

reflexive: Since $a = 1 \cdot a$ for all $a \in \mathbb{N}$, then $a \mid a$ and $a\mathcal{R}a$. Hence, $\mathcal{R}$ is reflexive.

antisymmetric: Assume that $a \mid b$ and $b \mid a$. Then, there are $h, k \in \mathbb{N}$ such that $b = ha$ and $a = kb$. Thus, $b = ha = h(kb) = (hk)b$. Then, $hk = 1$ which implies that $h = k = 1$. Therefore, $a = b$ and $\mathcal{R}$ is antisymmetric.

transitive: Assume that $a \mid b$ and $b \mid c$. Then, Theorem 1.4.1 implies that $a \mid c$. Thus, $a\mathcal{R}c$

and $\mathcal{R}$ is transitive. Therefore, $\mathcal{R}$ is a partial order on $\mathbb{N}$.

Example 3.4.3

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $a\mathcal{R}b$ iff $2 \mid a+b$ with $a \leq b$ for all $a, b \in \mathbb{N}$. Show that $\mathbb{N}$ is a poset with respect to $\mathcal{R}$.

Solution:

reflexive: Since $2 \mid a+a = 2a$ with $a \leq a$, $a\mathcal{R}a$ and $\mathcal{R}$ is reflexive.

antisymmetric: Assume that $a\mathcal{R}b$ and $b\mathcal{R}a$. Then, $2 \mid a+b$ with $a \leq b$ and $2 \mid b+a$ with $b \leq a$. Thus, $a \leq b \leq a$ which implies that $a = b$. Thus, $\mathcal{R}$ is antisymmetric.

transitive: Assume that $a\mathcal{R}b$ and $b\mathcal{R}c$. Then, $2 \mid a+b$ with $a \leq b$ and $2 \mid b+c$ with $b \leq c$. Therefore, by Theorem 1.4.1, $2 \mid a+2b+c$ which implies that $2 \mid a+c$ with $a \leq b \leq c$. Thus, $a\mathcal{R}c$ and $\mathcal{R}$ is transitive. Therefore, $\mathbb{N}$ is a poset with respect to $\mathcal{R}$.

3.4.1 Upper and Lower Bounds

Definition 3.4.3

Let $\mathcal{R}$ be a partial order for A and let B be any subset of A. Then,

- $a \in A$ is an **upper bound** for B if for every $b \in B$, $b\mathcal{R}a$. Also, a is called a "**least upper bound**" or "**supremum** for B, denoted by $\sup(B)$, if:
  1. a is an upper bound for B, and
  2. $a\mathcal{R}x$ for every upper bound x for B.
- $a \in A$ is a **lower bound** for B if for every $b \in B$, $a\mathcal{R}b$. Also, a is called a "**greatest upper bound**" or "**infimum** for B, denoted by $\inf(B)$, if:
  1. a is a lower bound for B, and
  2. $x\mathcal{R}a$ for every lower bound x for B.

Theorem 3.4.1

If $\mathcal{R}$ is a partial order for a set A and $B \subseteq A$, then if the least upper bound (or greatest lower bound) for B exists, then it is unique.

Proof:

Assume that x and y are both least upper bound for B. Since x is an upper bound and y is the least upper bound, thus $y\mathcal{R}x$. Similarly, since y is an upper bound and x is the least upper bound, thus $x\mathcal{R}y$. Since $\mathcal{R}$ is antisymmetric, $x\mathcal{R}y$ and $y\mathcal{R}x$, implies $x = y$.

Example 3.4.4

Let $A = [0, 6) \subset \mathbb{R}$ be a poset with respect to "$\leq$", and let $B = \{\frac{1}{2}, 3, 5\}$ and $C = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ be two subsets of A. Find $\sup(B)$, $\inf(B)$, $\sup(C)$, and $\inf(C)$.

Solution:

$\underline{\sup(B)}$: Note that $5, 5.1, 5.35, 5.9$, and so on are all considered upper bounds for B since for example $b \leq 5$ for all $b \in B$. Then, $\sup(B) = 5$ since $5 \leq x$ for all upper bounds for B.
$\underline{\inf(B)}$: $0, \frac{1}{2}, \frac{1}{4}, \frac{1}{45}$ and so on are all considered lower bounds for B since for example $\frac{1}{4} \leq b$ for all $b \in B$. Then, $\inf(B) = \frac{1}{2}$ since $\frac{1}{2} \leq x$ for all lower bounds x for B.
$\underline{\sup(C)}$: The set of upper bounds for C consists of $\{1, 2, 1.5, 3, 5, 5.5, \cdots\}$ while the $\sup(C) = 1$.
$\underline{\inf(C)}$: The set of upper bounds for C consists of $\{0\}$ and the $\inf(C) = 0$.
Note that, if $A = (0, 6)$, then C would has no $\inf(C)$.

Example 3.4.5

Let $A = \{1, 2, 3, 4, 5, 6\}$ and consider $\mathcal{P}(A)$ with the partial ordering "$\subseteq$". Let $B = \big\{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 6\}\big\}$. Find $\sup(B)$ and $\inf(B)$.

Solution:

Upper bound for B are like $\{1, 2, 3, 6\}$, $\{1, 2, 3, 4, 6\}$, $\{1, 2, 3, 5, 6\}$, and A it self. Therefore, $\sup(B) = \{1, 2, 3, 6\} = \bigcup_{X \in B} X$. On the other hand, ϕ, $\{1\}$, $\{2\}$, and $\{1, 2\}$ are all lower bounds for B while the $\inf(B) = \{1, 2\} = \bigcap_{X \in B} X$.

Exercise 3.4.1

Let $\mathcal{R}$ be a relation on $\mathbb{N}$ so that $x\mathcal{R}y$ iff $y = 2^k x$ for some integer $k \geq 0$. Show that $\mathbb{N}$ is a poset with respect to $\mathcal{R}$.