Chapter 4

Functions

Section 4.1: Functions as Relations

Definition 4.1.1

A **function** f from A to B is a relation from A to B that satisfies

1. $\mathrm{Dom}(f) = A$,
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Moreover, if $A = B$, we say that f is a function on A.

Remark 4.1.1: Notations

A function (mapping) f from A to B is denoted by $f : A \to B$. The **domain** of f is A and the **codomain** of f is B.

If $(x, y) \in f$, then $y = f(x)$ where we say that y is the **image** of x and that x is the **preimage** of y. The **range** of f is a subset of B and is defined as

$$\mathrm{Rng}(f) = \{y \in B : \exists x \in A \text{ with } y = f(x)\}.$$

Example 4.1.1

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Let $\mathcal{R}_1 = \{(1, a), (2, b), (2, c), (3, c)\}$, $\mathcal{R}_2 = \{(1, a), (2, c), (3, b)\}$, and $\mathcal{R}_3 = \{(1, a), (2, c)\}$ be three relations on $A \times B$. Decide whether $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ a function.

Solution:

$\mathcal{R}_1$ is clearly not a function since $(2, b)$ and $(2, c)$ both are in $\mathcal{R}_1$ where $b \neq c$. $\mathcal{R}_2$ satisfies the conditions of Definition 4.1.1 and so it is a function from A to B.

$\mathcal{R}_3$ is not a function from A to B; however, it is a function from $\{1, 2\}$ to $\{a, c\}$.

Example 4.1.2

Let $\mathcal{S} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ be a relation on $\mathbb{R}$. Is $\mathcal{S}$ a function? Explain.

Solution:

Note that for $x = 0$, we have $y = -1$ or $y = 1$ and so $\mathcal{S}$ is not a function. Another reason is that for $x = 5$, $y^2 = -24 \notin \mathbb{R}$.

Example 4.1.3

Let $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y = x^2\}$. Determine whether f a function on $\mathbb{Z}$.

Solution:

$f : \mathbb{Z} \to \mathbb{Z}$ is a function with $\mathrm{Rng}(f) = \{0, 1, 4, 9, 16, \cdots\}$. That is $f(x) = x^2$ is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

$\star$ **Constant Function**: $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = c$ (c is a constant) for all $x \in \mathbb{R}$.

Example 4.1.4

Let $f = \{\,(\,x, y\,) \in \mathbb{R} \times \mathbb{R} : y = 2x + 5\,\}$. Show that f is a function from $\mathbb{R}$ to $\mathbb{R}$.

Solution:

We first show that $\mathrm{Dom}(f) = \mathbb{R}$. Clearly, $\mathrm{Dom}(f) \subseteq \mathbb{R}$ by the definition of f. Next, let $x \in \mathbb{R}$. Then there is $y = 2x + 5 \in \mathbb{R}$ and hence $(x, y) \in f$. That is $x \in \mathrm{Dom}(f)$.
Now assume that $(x, y), (x, z) \in f$, we want to show that $y = z$. But since $y = 2x + 5$ and $z = 2x + 5$, we have $y = z$. Therefore, f is a function from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 4.1.1

Two functions f and g are equal iff (i) $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, and (ii) for all $x \in \mathrm{Dom}(f)$, $f(x) = g(x)$.

Proof:

„ $\Rightarrow$ ": Assume that $f = g$. Proof of (i): If $x \in \mathrm{Dom}(f)$, then $(x, y) \in f = g$ for some y and hence $x \in \mathrm{Dom}(g)$. Thus, $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(g)$. Similarly, if $x \in \mathrm{Dom}(g)$, then $(x, y) \in g = f$

for some y and hence $x \in \text{Dom}(f)$. Thus, $\text{Dom}(g) \subseteq \text{Dom}(f)$. Therefore, $\text{Dom}(f) = \text{Dom}(g)$. Proof of (ii): Let $x \in \text{Dom}(f)$. Then for some y, $(x, y) \in f = g$. Thus, $f(x) = y = g(x)$.

„ $\Leftarrow$ ": Assume that $\text{Dom}(f) = \text{Dom}(g)$ and that for all $x \in \text{Dom}(f)$, $f(x) = g(x)$. Suppose that $(x, y) \in f$, then there is y such that $y = f(x)$ and $x \in \text{Dom}(f) = \text{Dom}(g)$. Thus, $y = f(x) = g(x)$ which implies that $(x, y) \in g$ and hence $f \subseteq g$. Now suppose that $(x, y) \in g$. Then there is y such that $y = g(x) = f(x)$ for $x \in \text{Dom}(f)$. Thus, $y = f(x)$ and $(x, y) \in f$. Hence $g \subseteq f$. Therefore, $f = g$.

Section 4.2: Constructions of Functions

Definition 4.2.1

Let $f : A \to B$ and $g : B \to C$ be two given functions. The **composition function** $g \circ f$ is defined by $g \circ f : A \to C$ where $(g \circ f)(x) = g(f(x))$ for every $x \in A$. Note that $f \circ g \neq g \circ f$, while $(f \circ g) \circ h = f \circ (g \circ h)$ for any three (appropriate) functions f, g, and h.

Example 4.2.1

Let $f(x) = \sin(x)$ and $g(x) = 2x + 1$ for $x \in \mathbb{R}$. Find $f \circ g$ and $g \circ f$.

Solution:

For any $x \in \mathbb{R}$, we have

1. $(f \circ g)(x) = f(g(x)) = f(2x + 1) = \sin(2x + 1)$.

2. $(g \circ f)(x) = g(f(x)) = g(\sin(x)) = 2\sin(x) + 1$.

Definition 4.2.2

Let $f : A \to B$ and let $D \subseteq A$. The "**restriction of** f **to** D", denoted by $f|_D$, is a function with domain D and is defined as

$$f|_D = \{(x, y) : (x, y) \in f \text{ and } x \in D\}.$$

In that case, we say that f is an **extension** of $f|_D$.

Example 4.2.2

Let $f : A \to B$ be a function where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $f = \{(1, a), (2, a), (3, b), (4, c)\}$. Find $f|_A$, $f|_{\{1\}}$, and $f|_{\{2,4\}}$.

Solution:

Clearly, $f|_A = f$, $f|_{\{1\}} = \{(1, a)\}$, and $f|_{\{2,4\}} = \{(2, a), (4, c)\}$.

Remark 4.2.1

Let $f : A \to B$ and $g : C \to D$ be two functions. Then,

1. $f \cap g$ is a function with $\text{Dom}(f \cap g) = \{x \in A \cap C : f(x) = y = g(x) \in B \cap D\}$.

2. If $A \cap C = \phi$, then $f \cup g$ is a function with domain $A \cup B$.

Example 4.2.3

Let $f = \{(1,2),(3,5),(4,2)\}$ and $g = \{(1,2),(3,6),(5,-10)\}$. Find $f \cap g$ and $f \cup g$ and decide whether either of those relation is a function.

Solution:

Clearly, f is a function from $A = \{1,3,4\}$ to $B = \{2,5\}$ while g is a function from $C = \{1,3,5\}$ to $D = \{2,6,-10\}$. So,

- $f \cap g = \{(1,2)\}$ which is clearly a function from $\text{Dom}(f \cap g) = \{1\}$ to $\{2\}$.

- $f \cup g = \{(1,2),(3,5),(4,2),(3,6),(5,-10)\}$ which is not a function (by the definition) since 3 maps to two different values, namely 5 and 6.