Section 4.3: Functions That are Onto; One-to-One Functions

Definition 4.3.1

A function $f : A \to B$ is **onto (surjective mapping)** B iff $\operatorname{Rng}(f) = B$. Also, f is called a **surjection**. In that case, we write $f : A \xrightarrow{onto} B$.

Remark 4.3.1

Since $\operatorname{Rng}(f) \subseteq B$ is always true, f is a surjection iff $B \subseteq \operatorname{Rng}(f)$. Thus,

$$f : A \xrightarrow{onto} B \Longleftrightarrow (\forall b \in B)(\exists a \in A)(f(a) = b).$$

Example 4.3.1

Let $f(x) = x + 2$ and $g(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Determine whether f and g are onto $\mathbb{R}$.

Solution:

- f is onto: Let $y \in \mathbb{R}$ (in the range of f), then there exists $x \in \mathbb{R}$ such that $y = x + 2$ or $x = y - 2$. Thus, $f(x) = f(y - 2) = (y - 2) + 2 = y$. Thus, f is onto $\mathbb{R}$.
- g is not onto: Let $y \in \mathbb{R}$, then $y = x^2 + 1$ so $x = \pm\sqrt{y - 1}$. So, y must be greater than or equal to 1. If we choose $y = 0$, then $x \notin \mathbb{R}$ and hence g is not onto $\mathbb{R}$.

Example 4.3.2

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined by $f(m, n) = 2^{m-1}(2n - 1)$. Show that f is onto $\mathbb{N}$.

Solution:

We show that $\mathbb{N} \subseteq \operatorname{Rng}(f)$. That is, for all $s \in \mathbb{N}$, there exists $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $f(m, n) = s$. We consider the following two cases of s.

(i) if s is even: s can be written as $2^k t$, where $k \geq 1$ and t is odd. Since t is odd, $t = 2n - 1$ or $n = \frac{t+1}{2}$ for some $n \in \mathbb{N}$. Choosing $m = k + 1$, we have

$$f(m, n) = 2^{m-1}(2n - 1) = 2^k t = s.$$

Thus, $\mathbb{N} \subseteq \operatorname{Rng}(f)$.

(ii) if s is odd: $s = 2n-1$ for some $n \in \mathbb{N}$. Choosing $m = 1$, we have $f(m,n) = 2^0(2n-1) = s$. Thus, $\mathbb{N} \subseteq \text{Rng}(f)$.

Therefore, f is onto $\mathbb{N}$.

Theorem 4.3.1

Let A, B, and C be three sets. Then,

1. If $f : A \xrightarrow{onto} B$ and $g : B \xrightarrow{onto} C$, then $g \circ f : A \xrightarrow{onto} C$. That is, the composite of surjective functions is a surjection.

2. If $f : A \to B$, $g : B \to C$, and $g \circ f : A \xrightarrow{onto} C$, then g is onto C.

Proof:

1. We show that for every $c \in C$, $c \in \text{Rng}(g \circ f)$. Since g is onto C, there exists $b \in B$ such that $g(b) = c$. but since f is onto B, there exists $a \in A$ such that $f(a) = b$. Thus, $(g \circ f)(a) = g(f(a)) = g(b) = c$. Thus, $c \in \text{Rng}(g \circ f)$.

2. We show that again $C \subseteq \text{Rng}(g \circ f)$. Let $c \in C$. Since $g \circ f$ is onto C, there exists $a \in A$ such that $(g \circ f)(a) = c$. Let $b = f(a) \in B$. Then, $(g \circ f)(a) = g(f(a)) = g(b) = c$. Thus, there exists $b \in B$ such that $g(b) = c$ and hence g is onto.

Definition 4.3.2

A function $f : A \to B$ is said to be "**one-to-one**" (**injective mapping**) iff $(a_1, b) \in f$ and $(a_2, b) \in f$ imply that $a_1 = a_2$. Also, f is called an **injection**. In that case, we write $f : A \xrightarrow{1-1} B$.

Remark 4.3.2

A function $f : A \xrightarrow{1-1} B$ is one-to-one if and only if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \quad \text{or equivalently} \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2).$$

Example 4.3.3

Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x - 1$. Show that f is one-to-one.

Solution:

Assume that $f(a) = f(b)$, then $5a - 1 = 5b - 1 \Rightarrow 5a = 5b \Rightarrow a = b$. Thus, f is 1-1.

Example 4.3.4

Determine whether $f : \mathbb{R} \to \mathbb{R}$ is one-to-one, where $f(x) = \dfrac{1}{x^2+1}$.

Solution:

Assume that $f(a) = f(b)$, then

$$\frac{1}{a^2+1} = \frac{1}{b^2+1} \Rightarrow a^2 + 1 = b^2 + 1 \Rightarrow a^2 = b^2 \Rightarrow a = \pm b.$$

Therefore, f is not 1-1. For instance, $f(1) = f(-1)$ while $1 \neq -1$.

Example 4.3.5

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^{m-1}(2n - 1)$. Show that f is one-to-one.

Solution:

Assume that $f(a, b) = f(x, y)$ for $(a, b), (x, y) \in \mathbb{N} \times \mathbb{N}$. Then, $2^{a-1}(2b - 1) = 2^{x-1}(2y - 1)$. Consider the following cases:

1. if $a > x$: $2^{a-1}(2b-1) = 2^{x-1}(2y-1) \Rightarrow \underbrace{2^{a-x}(2b-1)}_{\text{even}} = \underbrace{(2y-1)}_{\text{odd}}$ which is impossible.
2. if $a < x$: $2^{a-1}(2b-1) = 2^{x-1}(2y-1) \Rightarrow \underbrace{(2b-1)}_{\text{odd}} = \underbrace{2^{x-a}(2y-1)}_{\text{even}}$ which is impossible.
3. if $a = x$: $2^{a-1}(2b-1) = 2^{x-1}(2y-1) \Rightarrow (2b-1) = (2y-1) \Rightarrow b = y$.

Thus, the only possible case is the third case which suggests that $(a, b) = (x, y)$. Therefore, f is 1-1.

Theorem 4.3.2

Let A, B, and C be three sets. Then,

1. If $f : A \xrightarrow{1-1} B$ and $g : B \xrightarrow{1-1} C$, then $g \circ f : A \xrightarrow{1-1} C$.
2. If $f : A \to B$ and $g : B \to C$, and $g \circ f : A \xrightarrow{1-1} C$, then $f : A \xrightarrow{1-1} B$.

Proof:

1. Assume that $(g \circ f)(x) = (g \circ f)(y)$ for some $x, y \in A$. Then, $g(f(x)) = g(f(y))$. Since, g is 1-1, $f(x) = f(y)$, and since f is 1-1 as well, $x = y$. Therefore, $g \circ f$ is 1-1.
2. Assume that $f(x) = f(y)$ for $x, y \in A$. Then $g(f(x)) = g(f(y))$ implies that $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f$ is 1-1, $x = y$. Thus, f is 1-1.

Remark 4.3.3

Horizontal Line Test: Let $f : A \to B$ be a given function. Then,

1. f is onto B iff for all $b \in B$, the horizontal line $y = b$ intersects the graph of f at least once.
2. f is one-to-one iff for all $b \in B$, the horizontal line $y = b$ intersects the graph of f at most once.

Example 4.3.6

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two function. Use the Horizontal line test to decide whether $f(x) = x^2$ and $g(x) = x^3$ are onto, one-to-one, or neither.

Solution:

We apply the horizontal line test on both f and g. In f, we see that on one place the line crosses the curve in two points, so f is not one-to-one, and it does not cross the curve in another place so it is not onto. However, in g, the line crosses the curve exactly once in any place, so it is one-to-one and onto.