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Lectures Notes in Functinoal Analysis

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Chapter I

Normed vector spaces, Banach spaces and metric spaces

1 Normed vector spaces and Banach spaces

In the following let X be a linear space (vector space) over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 1.1. *A* ***seminorm*** *on X is a map $p : X \to \mathbb{R}_+ = [0, \infty)$ s.t.*

(a) $p(\alpha x) = |\alpha| p(x) \quad \forall \alpha \in \mathbb{F}, \forall x \in X$ *(homogeneity).*

(b) $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X$ *(triangle inequality).*

If, in addition, one has

(c) $p(x) = 0 \Rightarrow x = 0$

then p is called a ***norm****. Usually one writes $p(x) = \|x\|$, $p = \|\cdot\|$. The pair $(X, \|\cdot\|)$ is called a* ***normed (vector) space****.*

Remark 1.2. • *If $\|\cdot\|$ is a seminorm on X then*

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \quad \forall x, y \in X \quad \textit{(reverse triangle inequality).}$$

Proof.

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$
$$\Rightarrow \|x\| - \|y\| \leq \|x - y\|$$

Now swap x&y: $\|x\| - \|y\| \leq \|y - x\| = \|(-1)(x - y)\| = \|x - y\|$.
Hence

$$\big|\|x\| - \|y\|\big| = max(\|x\| - \|y\|, \|y\| - \|x\|) \leq \|x - y\|.$$

□

- $\|\vec{0}\| = \|0.\vec{0}\| = |0|.\|\vec{0}\| = 0.$

Interpret $\|x - y\|$ as distance between x and y.

Definition 1.3. *Let $(x_n)_{n\in\mathbb{N}} = (x_n)_n$ be a sequence in a normed vector space X $((X, \|\cdot\|))$. Then $(x_n)_n$* ***converges*** *to a limit $x \in X$ if $\forall\varepsilon > 0 \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$ it holds $\|x_n - x\| < \varepsilon$ (or $\|x_n - x\| \leq \varepsilon$). One writes $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.*
$(x_n)_n$ is a ***Cauchy sequence*** *if $\forall\varepsilon > 0 \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n, m \geq N_\varepsilon$ it holds $\|x_n - x_m\| < \varepsilon$ (or $\|x_n - x_m\| \leq \varepsilon$).*
$(X, \|\cdot\|)$ is ***complete*** *if every Cauchy sequence converges.*
A complete normed space $(X, \|\cdot\|)$ is called a ***Banach space****.*

Remark 1.4. *Let X be a normed vector space, $(x_n)_n$ a sequence in X.*

(a) If $x_n \to x$ in X, then $(x_n)_n$ is a Cauchy sequence.

Proof. Given $\varepsilon > 0 \exists N_\varepsilon : \forall n \geq N_\varepsilon : \|x - x_n\| < \frac{\varepsilon}{2}$. Hence for $n, m \geq N_\varepsilon$ we have

$$\|x_n - x_m\| = \|x_n - x + x - x_m\| \leq \|x_n - x\| + \|x - x_m\| < \varepsilon.$$

□

(b) Limits are unique!
If $x_n \to x$ in X and $x_n \to y$ in X, then $x = y$.

Proof.

$$\begin{aligned}\|x - y\| &= \|x - x_n + x_n - y\| \\ &\leq \|x_n - x\| + \|x_n - y\| \to 0 + 0 = 0 \quad \text{as } n \to \infty.\end{aligned}$$

□

(c) If $(x_n)_n$ converges or is Cauchy, then it is bounded, i.e.

$$\sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Proof. Take $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ s.t. $\forall n, m \geq N : \|x_n - x_m\| < 1$. In particular, $\forall n \geq N : \|x_n - x_N\| < 1$.

$$\Rightarrow \|x_n\| = \|x_n - x_N + x_N\| \leq \|x_n - x_N\| + \|x_N\| < 1 + \|x_N\| \quad (\forall n \in \mathbb{N})$$

$$\Rightarrow \forall n \in N : \|x_n\| \leq max(\|x_1\|, \|x_2\|, \dots, \|x_N\|, 1 + \|x_N\|) < \infty.$$

□

Let X be a normed vector space, $S \neq \emptyset$ a set. For functions $f, g : S \to X$, $\alpha, \beta \in \mathbb{F}$ define

$$f + g : \begin{cases} S \to X \\ s \mapsto (f+g)(s) = f(s) + g(s) \end{cases}$$
$$\alpha f : \begin{cases} S \to X \\ s \mapsto (\alpha f)(s) = \alpha f(s) \end{cases}$$

So the set of functions from S to X is a normed space itself!

In case $X = \mathbb{R}$, we write $f \geq \alpha$ (or $f > \alpha$) if $f(s) \geq \alpha$ for all $s \in S$ ($f(s) > \alpha$ for all $s \in S$). Similarly one defines $\alpha \leq f \leq \beta$, $f \leq g$, etc.

Example 1.5. *(a) $X = \mathbb{F}^d, d \in \mathbb{N}$ is a Banach space (or short B-space) with respect to (w.r.t) the norms*

$$|x|_p := \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$
$$|x|_\infty := \max_{j=1,\dots,d} |x_j|.$$

Here $x = (x_1, x_2, \dots, x_d) \in \mathbb{F}^d$.

(b) Let $\Omega \neq \emptyset$, $L^\infty(\Omega) = L^\infty(\Omega, \mathbb{R}) =$ the set of all real-valued functions on Ω which are bounded, i.e.

$$f \in L^\infty(\Omega) \text{ then } \exists M_f < \infty : |f(\omega)| \leq M_f \text{ for all } \omega \in \Omega.$$

Norm on $L^\infty(\Omega)$: for $f \in L^\infty(\Omega)$: $\|f\|_\infty = \sup_{\omega\in\Omega} |f(\omega)|$ (check that this is a norm!).
Claim: $(L^\infty(\Omega), \|\cdot\|_\infty)$ is a Banach space.

Proof. Normed vector space is clear.
Take (f_n) a Cauchy sequence in $L^\infty(\Omega)$ w.r.t. $\|\cdot\|_\infty$. We have: $\forall \varepsilon > 0 \exists N : \forall n, m \geq N : \|f_n - f_m\|_\infty < \varepsilon$.
Fix $\omega \in \Omega$, then $\big(f_n(\omega)\big)$ is a Cauchy sequence in $\mathbb{R}$ since

$$|f_n(\omega) - f_m(\omega)| \leq \sup_{\omega\in\Omega} |f_n(\omega) - f_m(\omega)| = \|f_n - f_m\|_\infty < \varepsilon \quad \forall n, m \geq N.$$

Since $\mathbb{R}$ is complete, $f(\omega) := \lim_{n\to\infty} f_n(\omega)$ exists (this f is the candidate for the limit). We have

$$\begin{aligned} |f(\omega)| &\leq |f(\omega) - f_n(\omega)| + |f_n(\omega)| \\ &= \lim_{m\to\infty} |f_m(\omega) - f_n(\omega)| + |f_n(\omega)| \leq \varepsilon + \underbrace{|f_n(\omega)|}_{<\infty} \end{aligned}$$

$$\Rightarrow sup_{\omega\in\Omega} |f(\omega)| \leq \infty,$$

i.e., $f \in L^\infty(\Omega)$.
Take $\varepsilon > 0$. Then

$$|f_n(\omega) - f(\omega)| = \lim_{m\to\infty} \underbrace{|f_n(\omega) - f_m(\omega)|}_{\leq\varepsilon \text{ if } n,m\geq N} \leq \varepsilon \quad \text{if } n \geq N$$

$$\Rightarrow \forall n \geq N : \|f_n - f\|_\infty \leq \varepsilon, \text{ i.e., } f_n \to f \text{ w.r.t. } \|\cdot\|_\infty.$$

□

(c) $X = C([0,1]), \|f\|_1 := \int_0^1 |f(t)|dt$ *is a norm,* $\left(C([0,1]), \|\cdot\|_1\right)$ *is not complete.*

Proof.

$$\|f\|_1 \geq 0 \quad \forall f \in C([0,1]).$$

$$\|f+g\|_1 = \int_0^1 \underbrace{|f(t)+g(t)|}_{\leq |f(t)|+|g(t)|} dt \leq \|f\|_1 + \|g\|_1$$

$$\|\alpha f\|_1 = \int_0^1 |\alpha f(t)| dt = |\alpha| \|f\|_1$$

So $\|\cdot\|$ is a seminorm.
If $f \not\equiv 0$ and f is continuous, we see that there exist an interval $I \subset [0,1]$, $\delta > 0$ such that $|f(t)| \geq \delta \ \forall t \in I$.

$$\Rightarrow \|f\|_1 = \int_0^1 |f(t)| dt \geq \int_I \underbrace{|f(t)|}_{\geq \delta} dt \geq \delta.\text{lehgth of } I > 0.$$

So $\|\cdot\|_1$ is a seminorm. Now take a special sequence

$$f_n(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1, & \text{if } \frac{1}{2} - \frac{1}{n} < t < \frac{1}{2} \\ 1, \text{if } t \geq \frac{1}{2} \end{cases} \qquad (n \geq 3)$$

For $m \geq n \geq 3$:

$$\|f_n - f_m\|_1 = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{n}} |f_n(t) - f_m(t)| dt \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty,$$

so (f_n) is a Cauchy sequence.
Assume that $f_n \to f \in C([0,1])$. Fix $\alpha \in \left[0, \frac{1}{2}\right), n : \frac{1}{2} - \frac{1}{n} \geq \alpha$

$$0 \leq \int_0^\alpha |f(t)| dt = \int_0^\alpha |f_n(t) - f(t)| dt$$

$$\leq \int_0^1 |f_n(t) - f(t)| dt = \|f_n - f\|_1 \to 0.$$

Hence $f(t) = 0$ for all $0 \leq t \leq \alpha$, all $0 \leq \alpha < \frac{1}{2}$

$$f(t) = 0 \quad \text{for all} 0 \leq t < \frac{1}{2}.$$

On the other hand

$$0 \le \int_{\frac{1}{2}}^{1} |f(t) - 1| dt = \int_{\frac{1}{2}}^{1} |f(t) - f_n(t)| dt \le \|f - f_n\|_1 \to 0 \quad \text{as } n \to \infty$$

Since f is continuous on $[0, 1]$, it follows that $f(t) = 1$ for $\frac{1}{2} \le t \le 1$.
So f cannot be continuous at $t = \frac{1}{2}$. A contradiction. $\square$

2 Basics of metric spaces

Definition 2.1. *Given a set $M \neq \emptyset$, a metric (or distance) d on M is a function $d : M \times M \to \mathbb{R}$ such that*

(a) $d(x, y) \ge 0 \ \forall x, y \in M$ and $d(x, y) = 0 \iff x = y$.

(b) $d(x, y) = d(y, x) \ \forall x, y \in M$ (symmetry).

(c) $d(x, y) \le d(x, z) + d(z, y) \ \forall x, y, z \in M$ (triangle inequality).

*The pair (M, d) is called a **metric space**. We often simply write M if it is clear what d is.*
*A sequence $(x_n)_n$ in a metric space (M, d) **converges** to $x \in M$ if $\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \ge N_\varepsilon : d(x, x_n) < \varepsilon$ (or $\le \varepsilon$). One writes $\lim x_n = x$ or $x_n \to x$.*

One always has

$$\big|d(x, z) - d(z, y)\big| \le d(x, y)$$

Hint for the proof:

$$d(x, z) \le d(x, y) + d(y, z)$$

and think and use symmetry.

Example 2.2.

- *$\mathbb{R}$ with $d(x, y) = |x - y|$;*
- *Any normed vector space $(X, \|\cdot\|)$ with $d(x, y) = \|x - y\|$;*
- *Eucledian space $\mathbb{R}^d$ (or $\mathbb{C}^d$) with $d_2(x, y) = \big(\sum_{j=1}^{d} |x_j - y_j|^2\big)^{\frac{1}{2}}$ or $d_p(x, y) = \big(\sum_{j=1}^{d} |x_j - y_j|^p\big)^{\frac{1}{p}}$, or $d_\infty(x, y) = \max_{j=1,\dots d} |x_j - y_j|$.*
- *$M \neq \emptyset$, define $d : M \times M \to \mathbb{R}$ by*

$$d(x, y) := \begin{cases} 0, & \text{if } x = y \\ 1, & \text{else} \end{cases}$$

  is discrete metric. (M, d) is called discrete metric space.

- *$M = (0, \infty)$, $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ is a metric.*

- *Paris metric*

$$d(x,y) := \begin{cases} |x-y|, & \text{if } x = \lambda y \text{ for some } \lambda > 0, \\ |x| + |y|, & \text{else.} \end{cases}$$

- *If (M,d) is a metric space, $N \subset M$, then (N,d) is a metric space. Example: $M = \mathbb{R}^2, N = \{x : |x| = 1\}$.*

- *$M = \mathbb{F}^{\mathbb{N}}$ = set of all sequences $(a_n)_n, a_n \in \mathbb{F}$ = set of all functions $a : \mathbb{N} \to \mathbb{F}$ is a metric space with metric*

$$d(a,b) := \sum_{j=1}^{\infty} 2^{-j} \frac{|a(j)-b(j)|}{1+|a(j)-b(j)|}.$$

Proof. $d(a,b) \geq 0, d(a,b) = 0 \Rightarrow a = b, d(a,b) = d(b,a)$ are clear.
Need $d(a,b) \leq d(a,c) + d(c,b)$ for all sequences a, b, c.
Note: $0 \leq t \mapsto \frac{t}{1+t}$ is increasing!

$$\begin{aligned} d(a,b) &= \sum_{j=1}^{\infty} 2^{-j} \frac{|a(j)-b(j)|}{1+|a(j)-b(j)|} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \frac{|a(j)-c(j)| + |c(j)-b(j)|}{1+|a(j)-c(j)| + |c(j)-b(j)|} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \left(\frac{|a(j)-c(j)|}{1+|a(j)-c(j)|} + \frac{|c(j)-b(j)|}{1+|c(j)-b(j)|} \right), \end{aligned}$$

since, by the triangle inequality, $|a(j)-b(j)| \leq |a(j)-c(j)| + |c(j)-b(j)|$.
Note: $(a_n)_n \subset \mathbb{F}^{\mathbb{N}}, a_n \to a$ in $\mathbb{F}^{\mathbb{N}} \iff \forall j \in \mathbb{N} : a_n(j) \to a(j)$ and this space is complete!

$$\begin{aligned} 2^{-j} \frac{|a_n(j)-a(j)|}{1+|a_n(j)-a(j)|} &\leq d(a_n, a) \quad \text{for fixed } j \\ \Rightarrow |a_n(j)-a(j)| &\leq \underbrace{2^j d(a_n,a)}_{\leq \frac{1}{2} \text{ for } n \text{ large enough}} (1 + |a_n(j)-a(j)|) \\ &\leq 2^j d(a_n,a) + \frac{1}{2}|a_n(j)-a(j)| \text{ for } n \text{ large enough} \\ \Rightarrow \text{ for } n \text{ large enough:} &|a_n(j)-a(j)| \leq 2^{j+1} d(a_n,a) \to 0 \text{ as } n \to \infty, \end{aligned}$$

so $a_n \to a$ in $\mathbb{F}^{\mathbb{N}} \Rightarrow \forall j \in \mathbb{N} : a_n(j) \to a(j)$.
Need $\Leftarrow$:

$$\begin{aligned} d(a_n,a) &= \sum_{j=1}^{\infty} 2^{-j} \frac{|a_n(j)-a(j)|}{1+|a_n(j)-a(j)|} \\ &\leq \underbrace{\sum_{j=1}^{L} 2^{-j} |a_n(j)-a(j)|}_{\leq L \max_{j=1,\ldots,L} |a_n(j)-a(j)|} + \underbrace{\sum_{L+1}^{\infty} 2^{-j}}_{< \frac{\varepsilon}{2} \text{ by choosing} L \text{ large enough}} \end{aligned}$$

□

Definition 2.3. *Let (M, d) be a metric space.*

- *The* ***open ball*** *at x with radius $r > 0$: $B_r(x) := \{y \in M : d(x, y) < r\}$.*
- *$A \subset M$ is* ***open*** *if $\forall x \in A\ \exists r > 0$ with $B_r(x) \subset A$.*

Note: Every open ball is itself an open set! Indeed, $y \in B_r(x), r_1 := r - d(x, y) \Rightarrow B_{r_1}(x) \subset B_r(x)$ since, if $z \in B_{r_1}(y)$ then

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r_1 = r$$

so $z \in B_r(x)$.

Theorem 2.4. *(a) M and $\emptyset$ are open.*

(b) An arbitrary union of open sets is open.

(c) Finite intersections of open sets are open.

Proof. (a) Clear.

(b) Take $(A_j)_{j\in J}, A_j \subset M$ open.

$$x \in \bigcup_{j\in J} A_j = \{y \in M : \exists j \in J \text{ with } y \in A_j\} \Rightarrow \exists j \in J : x \in A_j.$$

Since A_j is open, there exists $r > 0$ with $B_r(x) \subset A_j \subset \bigcup_{j\in J} A_j$. Hence $\bigcup_{j\in J} A_j$ is open.

(c) Take $\{A_1, \dots A_n\}$ open sets in M

$$x \in A := \bigcap_{j=1}^{n} A_j = \{y \in M : y \in A_j \text{ for all } j = 1, \dots n\}$$

A_j open $\Rightarrow \exists r_j > 0 : B_{r_j}(x) \subset A_j, j = 1, \dots n$. Let $r := min(r_1, r_2, \dots, r_n) > 0$. Then

$$B_r(x) \subset B_{r_j}(x) \subset A_j \text{ for all } j = 1, \dots n$$

$$\Rightarrow B_r(x) \subset \bigcap_{j=1}^{n} A_j$$

□

Definition 2.5. *(a) $x \in A$ is called an* ***interior point*** *of A if $\exists r > 0 : B_r(x) \subset A$. The set of all interior points is denoted by A^o.*
Note:

- *A^o is the largest open subset of M contained in A.*
- *A is open $\iff A = A^o$.*

(b) $A \subset M$ is ***closed*** *if its complement $A^c := M \setminus A = \{x \in M : x \notin A\}$ is open;*

Theorem 2.6. *(a) M and $\emptyset$ are closed.*

(b) Arbitrary intersections of closed sets are closed.

(c) Finite unions of closed sets are closed.

Proof. (a) $M^c = \emptyset$, $\emptyset^c = M$ are open.

(b) $(A_j)_{j\in J}$ family of closed sets. By Theorem 2.4 and de Morgan's law

$$\Big(\bigcap_{j\in J} A_j\Big)^c = \bigcup_{j\in J} A_j^c \text{ is open,}$$

so $\bigcap_{j\in J} A_j$ is closed;

(c) Combine $\left(\bigcup_{j=1}^n A_j\right)^c = \bigcap_{j=1}^n A_j^c$ with (c) of Theorem 2.4.

□

Definition 2.7. *A point $x \in M$ is called **closure point** of $A \subset M$ if $\forall r > 0 :$ $B_r(x) \cap A \neq \emptyset$. The set of all closure points of A is denoted by $\overline{A}$ and it is called the **closure** of A.*
Clearly $A \subset \overline{A}$.

Theorem 2.8. *Let (M, d) be a metric space, $A \subset M$. Then $\overline{A}$ is the smallest closed set that contains A.*

Remark 2.9. *Let $\mathcal{F}_A := \{B \subset M : B$ is closed and $A \subset B\}$. Then the smallest closed subset of M that contains A is, of course, given by $\bigcap_{B\in\mathcal{F}_A} B$. (think about this!)*

Proof of Theorem 2.8. Let $A \subset M$.
Step 1: $\overline{A}$ is closed. Indeed, if $x \in (\overline{A})^c$, then $\exists r > 0$ with $B_r(x) \cap A = \emptyset$. We want to show that $B_r(x) \subset (\overline{A})^c$, because then $(\overline{A})^c$ is open, hence $\overline{A}$ is closed. Let $y \in B_r(x)$. Since $B_r(x)$ is open, there exists $\delta > 0$ with $B_\delta(y) \subset B_r(x)$

$$\Rightarrow B_\delta(y) \cap A \subset B_r(x) \cap A = \emptyset$$

$\Rightarrow y \notin \overline{A}$ and since $y \in B_r(x)$ was arbitrary, this shows

$$B_r(x) \cap \overline{A} = \emptyset$$

so $B_r(x) \subset (\overline{A})^c$, hence $(\overline{A})^c$ is open.
Step 2: Let $B \subset M$ be closed with $A \subset B$. We show $\overline{A} \subset B$. Indeed, take $x \in B^c$. Since B^c is open, there exists $r > 0$ with $B_r(x) \subset B^c$, i.e., $B_r(x) \cap B = \emptyset$. In particular, $B_r(x) \cap A \subset B_r(x) \cap B = \emptyset$. So no point in B^c is a closure point of $A \Rightarrow \overline{A} \subset (B^c)^c = B$. □

Corollary 2.10. *$A \subset M$ is closed $\Rightarrow A = \overline{A}$.*

Proof. Have a close look at Theorem 2.8. □

Remark 2.11. • *For $a \in M$ and $r > 0$ call*

$$B_{\overline{r}}(a) := \{x \in M : d(x,a) \leq r\}$$

the closed ball at a with radius r. This is always a closed set. Indeed, assume $x \notin B_{\overline{r}}(a)$, i.e., $d(x,a) > r$ and set $r_1 := d(x,a) - r > 0$. If $y \in B_{r_1}(x)$, then

$$d(a,x) \leq d(a,y) + d(y,x)$$

$$\Leftrightarrow d(a,y) \geq d(a,x) - d(y,x) > d(a,x) - r_1 = r,$$

i.e., $y \notin B_{\overline{r}}(a)$, hence $B_{r_1}(x) \subset (B_{\overline{r}}(a))^c$ so $(B_{\overline{r}}(a))^c$ is open $\Leftrightarrow$ $B_{\overline{r}}(a)$ is closed.

- *One always has $\overline{B_r(a)} \subset B_{\overline{r}}(a)$. In a discrete metric space the above inclusion can be strict! But, e.g., in $\mathbb{R}^d$ with the distance $d_p, 1 \leq p \leq \infty$, one always has $\overline{B_r(a)} = B_{\overline{r}}(a)$. (think about this!)*

Lemma 2.12. *If (M,d) is a metric space, then $A^o = (\overline{A^c})^c$.*

Proof.

$$\begin{aligned} x \in A^o &\Leftrightarrow \exists r > 0 : B_r(x) \subset A \\ &\Leftrightarrow B_r(x) \cap A^c = \emptyset \\ &\Leftrightarrow x \notin \overline{A^c} \\ &\Leftrightarrow x \in (\overline{A^c})^c. \end{aligned}$$

□

Definition 2.13. *Let (M,d) be a metric space, $A \subset M$. A point $x \in M$ is an* ***accumulation point*** *of A if*

$$\forall r > 0 \quad B_r(x) \cap (A \setminus \{x\}) \neq \emptyset,$$

i.e., every open ball around x contains an element of A different from x.

Note:

- It can be that $x \notin A$!
- Every accumulation point is a closure point of A.
- If one denotes the set of all accumulation points of A by A', then one has $\overline{A} = A \cup A'$ (why?).

Theorem 2.14. *Let $A \subset M$, (M,d) a metric space. Then $x \in M$ belongs to $\overline{A}$ if and only if (iff) there is a sequence $(x_n)_n \subset A$ with $\lim x_n = x$. Moreover, if x is an accumulation point of A, then there exists a sequence $(x_n)_n \subset A$ with $x \neq x_n \neq x_m, n \neq m$, i.e., all terms are distinct.*