جامعة الانبار

كلية التربية للعلوم الصرفة

قسم الرياضيات / ماجستير

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(المصدر)

Lectures Notes in Functinoal Analysis

WS 2012 – 2013

Proof. Let $x \in \overline{A}$. Given $n \in \mathbb{N}$ pick x_n with $x_n \in B_r(x) \cap A(\neq \emptyset$ since $x \in \overline{A}!)$. Then $x_n \in A$ and $\lim x_n = x$.
Conversely, if $x_n \in A$ and $\lim x_n = x$, then given $r > 0$ there exists $k \in \mathbb{N}$ such that $d(x, x_n) < r$ for all $n \geq k$. Therefore $B_r(x) \cap A \neq \emptyset$ for all $r > 0 \Rightarrow x \in \overline{A}$.
If x is an accumulation point of A, choose $x_1 \in A, x_1 \neq x$ and $d(x, x_1) < 1$. Then, inductively, if $x_1, \dots x_n \in A \setminus \{x\}$ pich $x_{n+1} \in A \setminus \{x\}$ with

$$d(x, x_{n+1}) < \min\Big(\frac{1}{n+1}, d(x, x_n)\Big).$$

Thus $(x_n)_n$ is a sequence in $A \setminus \{x\}$, $x_n \neq x_m$ if $n \neq m$ and $\lim x_n = x$. $\square$

Definition 2.15. *$A \subset M$ is **dense** in M if $\overline{A} = M$.*

Remark 2.16.
- *By Theorem 2.14, A is dense in M iff $\forall x \in M, \exists$ sequence $(x$ A with $\lim x_n = x$.*
- *A is dense in $M \Leftrightarrow V \cap A \neq \emptyset$ for every nonempty open set V.*

Definition 2.17. *Let $A \subset M$. $x \in M$ is a **boundary point** of A if $\forall r > 0 :$ $B_r(x) \cap A \neq \emptyset \neq B_r(x) \cap A^c$. The set of all boundary points of A is denoted by ∂A and it is called **boundary** of A.*

Note:

- By symmetry, $\partial A = \partial(A^c)$.
- $\partial A = \overline{A} \cap \overline{A^c}$ (Why?)

Definition 2.18 (Continuity)**.** *Let $(M, d), (N, \rho)$ be two metric spaces. A function $f : M \to N$ is*

- *continuous at a point $a \in M$ if $\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0$ with $\rho\big(f(x), f(a)\big) < \varepsilon$ for all $d(x, a) < \delta$.*
- *continuous on M (or simply continuous) if f is continuous at every point of M.*
- *sequentially continuous at a point $a \in M$ if for every sequence $(x_n)_n \subset M, x_n \to a$ one has $f(x_n) \to f(a)$.*
- *sequenctially continuous on M (or simply sequentially continuous) if it is sequentially continuous at every point of M.*
- *topologically continuous if for every open set $\mathcal{O}$ the set $f^{-1}(\mathcal{O}) \subset M$ is open.*

Theorem 2.19. *For a function $f : (M, d) \to (N, \rho)$ between two metric spaces, the following are equivalent:*

(a) f is continuous on M.

(b) f is topologically continuous on M.

(c) f is sequentially continuous on M.

(d) $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subset M$.

(e) $f^{-1}(\mathcal{C}) \subset M$ is closed for every closed subset $\mathcal{C} \subset N$.

Remark 2.20. *For a fixed $a \in M$, the following are also equaivalent:*

(a') f is continuous at a.

(c') f is sequentially continuous at a.

(Prove this!)

Proof of Theorem 2.19. $\underline{(a) \Rightarrow (b)}$: Let $\mathcal{O} \subset N$ be open and $a \in f^{-1}(\mathcal{O})$. Since $f(a) \in \mathcal{O}$ and $\mathcal{O}$ is open, there exists $r > 0$ such that $B_r(f(a)) \subset \mathcal{O} \subset N$. f continuous implies that there exists $\delta > 0$ such that

$$d(x, a) < \delta \Rightarrow \rho(f(x), f(a)) < r,$$

i.e., $B_\delta(a) \subset f^{-1}(\mathcal{O})$ and so $f^{-1}(\mathcal{O})$ is open.
$\underline{(b) \Rightarrow (c)}$: Let $x_n \to x$ in M and $\varepsilon > 0$. Let $V := B_\varepsilon(f(x)) \subset N$, which is open. Then $f^{-1}(V)$ is open in M and since $x \in f^{-1}(V)$ there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(V)$. Let $N \in \mathbb{N}$ be such that $n \geq N \Rightarrow x_n \in B_\delta(x)$ (i.e., $d(x_n, x) < \delta$ for all $n \geq N$), Then also $x_n \in f^{-1}(V)$, so $f(x_n) \in V$, i.e., $\rho(f(x_n), f(x)) < \varepsilon$ for all $n \geq N$. Thus $f(x_n) \to f(x)$.
$\underline{(c) \Rightarrow (d)}$: Let $A \subset M$. Assume $y \in f(\bar{A})$. Then there exists $x \in \bar{A}$ with $f(x) = y$. Since $x \in \overline{A}$, by Theorem 2.14, it follows that there exists a sequence $(x_n)_n \subset A$ with $x_n \to x$, but then by (c): $f(x_n) \to f(x)$ in N, i.e., $y \in \overline{f(A)}$. So $f(\bar{A}) \subset \overline{f(A)}$.
$\underline{(d) \Rightarrow (e)}$: Let $\mathcal{C} \subset N$ be closed, so $\bar{\mathcal{C}} = \mathcal{C}$. Let $A := f^{-1}(\mathcal{C})$. Then by (d) we have

$$f(\overline{A}) \subset \overline{f(A)} = \bar{\mathcal{C}} = \mathcal{C},$$

so $\bar{A} \subset f^{-1}(\mathcal{C}) = A$. Since $A \subset \bar{A}$ is always true, we must have $f^{-1}(\mathcal{C}) = A = \bar{A}$, i.e., $f^{-1}(\mathcal{C})$ is closed.
$\underline{(e) \Rightarrow (a)}$: Let $a \in M$ and $\varepsilon > 0$. Consider

$$\mathcal{C} := B_\varepsilon\big(f(a)\big)^c = \{y \in N : \rho(f(a), y) \geq \varepsilon\}$$

which is closed. By (e) $f^{-1}(\mathcal{C}) \subset M$ is closed, i.e., $\Big(f^{-1}(\mathcal{C})\Big)^c$ is open. Thus, since $a \notin f^{-1}(\mathcal{C})$, i.e., $a \in \Big(f^{-1}(\mathcal{C})\Big)^c$, there exists $\delta > 0$ such that $B_\delta(a) \subset \Big(f^{-1}(\mathcal{C})\Big)^c$. But then $d(x, a) < \delta \Rightarrow \rho(f(x), f(a)) < \varepsilon$, i.e., f is continuous. ☐

Remark 2.21. *It should be clear that compositions of continuous functions are continuous.*

Definition 2.22.
- *Two metric spaces $(M, d), (N, \rho)$ are **homeomorphic** if $\exists$ a one-to-one onto function (i.e., bijection) $f : (M, d) \to (N, \rho)$ such that both f and f^{-1} are continuous;*
- *Two metrics d and ρ on M are **equaivalent** if a sequence $(x_n)_n \subset M$ satisfies*

$$\lim d(x_n, x) = 0 \iff \lim \rho(x_n, x) = 0,$$

or equaivalently, if any open set w.r.t. d is open w.r.t. ρ and conversely.

- *A metric space M is* ***bounded****, if $\exists 0 < M < \infty$ s.t. $d(x,y) \leq M \forall x, y \in M$. The* ***diameter*** *of $A \subset M$ is*

$$d(A) := \sup(d(x,y) : x, y \in A).$$

Note: If d is a metric on M

$$\rho(x,y) := \frac{d(x,y)}{1 + d(x,y)}$$

is an equaivalent metric on M under which M is bounded!

- *A sequence $(x_n)_n$ in a metric space (M,d) is a* ***Cauchy sequence*** *if $\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} : d(x_n, x_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$.*

  Note: Every convergent sequence $(x_n)_n$ is a Cauchy sequence (Why?). The converse is not true, e.g. take $M = (0, \infty), d(x,y) = |x - y|$. Then $x_n = \frac{1}{n}$ is Cauchy but not convergent in M.

- *A metric space (M,d) is* ***complete*** *(or complete metric space) if every Cauchy sequence converges (in M).*

Example 2.23. • *$\mathbb{R}^d$ with Eucledean metric or with $d_p, 1 \leq p \leq \infty$.*

- *$L^\infty(S), S \neq \emptyset$, $D(f,g) := \sup_{x \in S} |f(s) - g(s)|$.*

Theorem 2.24. *Let (M,d) be a complete metric space. Then $A \subset M$ is closed if and only if (A,d) is a complete metric space (in its own right).*

Proof. "$\Rightarrow$": Let $A \subset M$ be closed, $(x_n)_n \subset A$ be Cauchy $\Rightarrow (x_n)_n$ is Cauchy in M. Since M is complete, it follows that $x = \lim_{n\to\infty} x_n$ exists in M. Since A is closed, we conclude that $x \in A$. So $(x_n)_n$ converges in A and thus (A,d) is complete.

"$\Leftarrow$": Let (A,d) be complete. Let $(x_n)_n \subset A$ converge to some $x \in M$. So $(x_n)_n$ is Cauchy in A, A is complete $\Rightarrow (x_n)_n$ converges to some point in $A \subset M$. The limit is unique so $x = \lim_{n\to\infty} x_n \in A$. So A is closed. □

Lemma 2.25. *Let (M,d) be a metric space and $(x_n)_n, (y_n)_n \subset M$ s.t. $x_n \to x, y_n \to y$. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = d(x,y).$$

Proof. By the triangle inequality one has

$$|d(x,z) - d(z,y)| \leq d(x,y)$$

$$\begin{aligned}\Rightarrow |d(x_n,y_n) - d(x,y)| &\leq |d(x_n,y_n) - d(x,y_n)| + |d(x,y_n) - d(x,y)| \\ &\leq d(x_n,x) + d(y_n,y) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

□

Definition 2.26. *A function $f : (M,d) \to (N,\rho)$ is called* ***uniformly continuous*** *if $\forall \varepsilon > 0 \exists \delta > 0 : x, y \in M, d(x,y) < \delta (or \leq \delta) \Rightarrow \rho(f(x), f(y)) < \varepsilon (or \leq \varepsilon)$.*

Remark 2.27. • *Every uniformly continuous function is continuous.*

- $M = (0,1], N = \mathbb{R}, d(x,y) = |x-y|, d : (0,1] \to \mathbb{R}, x \mapsto f(x) = x^2$ *is uniformly continuous,* $g : (0,1] \to \mathbb{R}, x \mapsto g(x) = \frac{1}{x}$ *is continuous but not uniformly continuous.*

Theorem 2.28. *Let A be a subset of a metric space (M,d), (N,ρ) be a complete metric space. If $f : A \to N$ is uniformly continuous, then f has a unique uniformly continuous extension to the closure $\overline{A}$ of A.*

Remark 2.29. *This does not hold if f is only continuous!*
Example: 1) $f : (0,1] \to \mathbb{R}, x \to \frac{1}{x}$.
2) $f : \mathbb{Q} \to \mathbb{R}, x \mapsto \begin{cases} 1, & \text{if } x^2 \geq 2 \\ -1, & \text{if } x^2 < 2 \end{cases}$ *is a continuous function in* $\mathbb{Q}$*!*
Note that f also is differentiable on $\mathbb{Q}$ with zero derivative!
Look at $g(x) = x + 4f(x), x \in \mathbb{Q} \Rightarrow g'(x) = 1$. So g "must" be increasing! (?). But

$$g(-2) = 2,$$
$$g(0) = -4.$$

So g is not increasing!

Proof of Theorem 2.28. Step 1: Uniqueness should be clear (why?).
Step 2: Let $x \in \overline{A}$. By Theorem 2.14, there exists a sequence $(x_n)_n \subset A$ with $x_n \to x$.
Claim: $\lim_{n\to\infty} f(x_n)$ exists in (N,ρ)!
(N,ρ) is complete $\Rightarrow$ we only need to show that $(f(x_n))$ is Cauchy in (N,ρ). Let $\varepsilon > 0$. Since f is uniformly continuous, $\exists \delta > 0 : d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$. So let $N_\varepsilon \in \mathbb{N}$ be such that $d(x_n, x_m) < \delta$ for all $n, m \geq N_\varepsilon \Rightarrow \rho(f(x_n), f(x_m)) < \varepsilon$ for all $n, m \geq N_\varepsilon$.
Step 3: The limit $\lim_{n\to\infty} f(x_n)$ in Step 2 is independent of the sequence as long as $x_n \to x$. Indeed, let $(x_n)_n, (y_n)_n \subset A, x_n \to x, y_n \to x$ in M. By Step 2 we know that $u = \lim f(x_n), v = \lim f(y_n)$ exist in N. We want to show $u = v$.
For $n \in \mathbb{N}$, let $z_{2n} = x_n, z_{2n-1} = y_n \Rightarrow z_n \to x$ also, and, by Step 2: $\lim f(z_n)$ exists. We have

$$v = \lim f(y_n) = \lim f(z_{2n-1}) = \lim f(z_n) = \lim f(z_{2n}) = \lim f(x_n) = u.$$

Step 4: Define $f^* = \lim f(x_n), x_n \in A, x_n \to x$ (well defined by Steps 2&3). Of course $f^*(x) = f(x), x \in A$ is an extension of A to $\overline{A}$.
Step 5: $f^* : \overline{A} \to N$ is uniformly continuous. Indeed, given $\varepsilon > 0$, let $\delta > 0$ such that $x, y \in A, d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$. Now if $x, y \in \overline{A}$ satisfy $d(x,y) < \delta$, let $(x_n)_n, (y_n)_n \subset A, x_n \to x, y_n \to y$. By Lemma 2.25

$$\lim d(x_n, y_n) = d(x,y) < \delta \Rightarrow \exists N_0 \in \mathbb{N} : d(x_n, y_n) < \delta \quad \text{or all } n \geq N_0.$$

Since f is uniformly continuous

$$\rho(f(x_n), f(y_n)) < \varepsilon$$

By Lemma 2.25

$$\rho(f(x), f(y)) = \lim \rho(f(x), f(y)) \leq \varepsilon$$

so f^* is uniformly continuous. ∎

Definition 2.30. • *A function $f : (M, d) \to (N, \rho)$ is an* ***isometry*** *if $\rho(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. If f is also onto, then (M, d) and (N, ρ) are isometric.*

Note: any isometry is uniformly continuous!

- *A complete metric space (N, ρ) is called a* ***completion*** *of a metric space (M, d) if there exists an isometry $f : (M, d) \to (N, \rho)$ such that $f(M) = \{y \in N : \exists x \in M : y = f(x)\}$ is dense in N (w.r.t ρ).*
  If we think of M and $f(M)$ as identical, then M can be considered to be a subset of N.

Remark 2.31. *Any two completions of a metric space (M, d) must be isometric.*

Proof. Indeed, if N_1, N_2 are completions of M:

$$N_2 \supset \text{dense } g(M) \xleftarrow{g} M \xrightarrow{f} f(M) \text{ dense} \subset N_1$$

f, g are isometries. Define $h := g \circ f^{-1} : f(M) \to f(M)$. h is also an isometry (so, it is uniformly continuous). $f(M)$ is dense in N_1, N_2 is complete, so by Theorem 2.23 h has a unique uniformly continuous extension $\tilde{h} : N_1 \to N_2$.
Note: $\tilde{h}$ is an isometry from N_1 onto N_2! (Why?) (use that $g(M)$ is dense in N_2). □

Our approach to completeness: Given a metric space (M, d), find a complete metric space (N, ρ) and an isometry $f : (M, d) \to (N, \rho)$. $f(M)$ is then isometric to M. Take the closure $\overline{f(M)}$ in N. Then $(\overline{f(M)}, \rho) \subset (N, \rho)$ is a completion of (M, d)!

Theorem 2.32. *Every metric space (M, d) has a unique (up to isometries) completion.*

Proof. Goal: Embedd M in a complete metric space and take the closure!
We will use $(L^\infty(M), D)$, the bounded real-valued functions on M with

$$D(f, g) := \sup_{x \in M} |f(x) - g(x)|.$$

Fix $a \in M$. For $x \in M$ let

$$f_x : \begin{cases} M \to \mathbb{R}, \\ y \mapsto f_x(y) := d(x, y) - d(y, a). \end{cases}$$

By the reverse triangle inequality:

$$|f_x(y)| = |d(x, y) - d(y, a)| \leq d(x, a)$$

So $f_x \in L^\infty(M)$. Hence there exists a unique

$$f : \begin{cases} M \to L^\infty(M), \\ x \mapsto f_x. \end{cases}$$

Claim: f is an isometry!
Indeed, for $x, y, z \in M$:

$$\begin{aligned} |f_x(y) - f_z(y)| &= |d(x, y) - d(y, a) - (d(z, y) - d(y, a))| \\ &= |d(x, y) - d(z, y)| \leq d(x, z). \end{aligned}$$

$$\Rightarrow D(f_x, f_y) = \sup_{y \in M} |f_x(y) - f_z(y)| \leq d(x, z).$$

Choose $y = z$:

$$|f_x(z) - f_z(z)| = |d(x, z) - d(z, z)| = d(x, z),$$

so

$$D(f_x, f_z) = d(x, z).$$

Since $(L^\infty(M), D)$ is a complete metric space $\Rightarrow$ $(\overline{f(M)}, D)$ is a completion of (M, d). □

3 Compactness in metric space

In the following let (M, d) be a metric space.

Definition 3.1 (Totally bounded set). *A subset $A \subset M$ is totally bounded if $\forall \varepsilon > 0\ \exists n \in \mathbb{N} : x_1, \dots, x_n \in M$ with $A \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$ (so each $x \in A$ is within ε-distance from some x_i).*

Remark 3.2. *(a) Every $x \in A$ can be approximated up to error ε by one of the x_i.*

(b) In a finite dimensional (vector) space totally bounded is equivalent to bounded. In general totally bounded $\Rightarrow$ bounded, but the converse in wrong!

(c) In Definition 3.1 we could easily insist that each ε-ball is centered at some point in A. Indeed, let $\varepsilon > 0$, choose $x_1, \dots, x_n \in M$.

$$A \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{2}}(x_i).$$

W.l.o.g., we may assume that $B_{\frac{\varepsilon}{2}}(x_i) \cap A \neq \emptyset$. Then choose any $y_i \in A \cap B_{\frac{\varepsilon}{2}}(x_i)$. By the triangle inequality: $B_{\frac{\varepsilon}{2}}(x_i) \subset B_\varepsilon(y_i) \Rightarrow A \subset \bigcup_{i=1}^n B_\varepsilon(y_i)$.

Lemma 3.3. *$A \subset M$ is totally bounded $\Leftrightarrow \forall \varepsilon > 0$ there exist finitely many sets $A_1, \dots A_n$ with $diam(A_i) < \varepsilon$ for all $i = 1, \dots n$ and $A \subset \bigcup_{i=1}^n A_i$.*

Proof. "$\Rightarrow$": Let A be totally bounded. Given $\varepsilon > 0$ choose $x_1, \dots x_n \in M$ with $A \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$. Let $A_i := A \cap B_\varepsilon(x_i)$ to see that $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n A \cap B_\varepsilon(x_i) = A \cap (\bigcup_{i=1}^n B_\varepsilon(x_i) = A$ and note that $diam(A_i) < 2\varepsilon$.
"$\Leftarrow$": Given $\varepsilon > 0$ assume that there are finitely many $A_i \subset A, i = 1, \dots, n$, $diam(A_i) < \varepsilon, A \subset \bigcup_{i=1}^n A_i$. Then choose any $x_i \in A_i \Rightarrow A_i \subset B_{2\varepsilon}(x_i)(\forall i = 1 \dots n) \Rightarrow A \subset \bigcup_{i=1}^n B_{2\varepsilon}(x_i)$. □

Remark 3.4. *In Lemma 3.3 we insisted on $A_i \subset A(\forall i = 1 \dots n)$. This is not a real constraint. If A is covered by $B_1, \dots, B_n \subset M$, $diam(B_i) < \varepsilon$. Then A is also covered by $A_i = A \cap B_i \subset A$ and $diam(A_i) \leq diam(B_i) < \varepsilon$.*

There is also a sequential criterion for total boundedness. The Key observation is

Lemma 3.5. *Let $(x_n)_n \subset M, A = \{x_n : n \in \mathbb{N}\}$. Then*

(a) If $(x_n)_n$ is Cauchy, then A is totally bounded.

(b) If A is totally bounded, then $(x_n)_n$ has a Cauchy subsequence.

Proof. (a) Let $\varepsilon > 0$. Since $(x_n)_n$ is Cauchy, there exists $N \in \mathbb{N}$ with

$$d(x_n, x_m) < \frac{\varepsilon}{2} \quad \text{for all } n, m \geq N$$

$$\Rightarrow \sup_{n,m \geq N} d(x_n, x_m) \leq \frac{\varepsilon}{2} < \varepsilon$$

$$\Rightarrow diam\{x_n : n \geq N\} = \sup_{n,m \geq N} d(x_n, x_m) < \varepsilon$$

$$\Rightarrow \{x_n : n \geq N\} \subset B_\varepsilon(x_N).$$

(b) If A is finite, we are done because by pidgeonholing, there must be a point in A which the sequence $(x_n)_n$ hits infinitely often. Thus $(x_n)_n$ even has a constant subsequence in this case.
So assume that A is an infinite totally bounded set. Then A can be covered by finitely many sets of diameter < 1. At least one of them must contain infinitely many points of A. Call this set A_1. Note that A_1 is totally bounded, so it can itself be covered by finitely many sets of diameter $< \frac{1}{2}$. One of these, call it A_2, contains infinitely many points of A_1. Continuing inductively we find a decreasing sequence of sets $A \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \ldots$ where each A_k contains infinitely many x_n and where $diam(A_k) < \frac{1}{k}$.
Now choose a subsequence $(x_{n_k})_k, x_{n_k} \in A_k, k \in \mathbb{N}$. This subsequence is Cauchy, since

$$\sup(d(x_{n_l}, x_{n_m})l, m \geq k) \leq diam(A_k) < \frac{1}{k}.$$

□

Theorem 3.6 (Sequential characterization of total boundedness). *A set $A \subset M$ is totally bounded $\iff$ every sequence in A has a Cauchy subsequence.*

Proof. "$\Rightarrow$": Clear by Lemma 3.5.
"$\Leftarrow$": Assume A is not totally bounded. So for some $\varepsilon > 0$, A cannot be covered by finitely many ε-balls. By induction, there is a sequence $(x_n)_n \subset A$ with $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$ (Why?). But this sequence has no Cauchy subsequence! □

Corollary 3.7 (Bolzano-Weierstraß). *Every bounded infinite subset of $\mathbb{R}^d$ has an accumulation point.*

Proof. Let $A \subset \mathbb{R}^d$ be bounded and infinite. Then there is a sequence $(x_n)_n$ of distinct points in A. Since A is totally bounded ($\mathbb{R}^d$ has dimension $d < \infty$) there is a Cauchy subsequence of $(x_n)_n$, but $\mathbb{R}^d$ is complete, so $(x_n)_n$ converges to some $x \in \mathbb{R}^d$. This x is an accumulation point of A. □

Now we come to compactness.

Definition 3.8.
- *A metric space (M, d) is compact if it is complete and totally bounded.*
- *A subset $A \subset M$ is compact, if (A, d) is a compact metric space.*

Example 3.9. *(a) $K \subset \mathbb{R}^d$ is compact $\iff$ K is closed and bounded.*

(b) Let l^∞ = set of all bounded sequences and let

$$e_n := \delta_n, \quad \delta_n(j) := \begin{cases} 1, & \text{if } j = n, \\ 0, & \text{else.} \end{cases}$$

Then the set $A := \{e_n | n \in \mathbb{N}\}$ is closed and bounded, but not totally bounded, since

$$d(e_n, e_m) = \sup_{j \in \mathbb{N}} |e_n(j) - e_m(j)| = 1, \quad \text{if } n \neq m,$$

hence, A cannot be covered by finitely many $\varepsilon = \frac{1}{2}$-balls! (Why?)

(c) A subset of a discrete metric space is compact $\iff$ A is finite. (Why?)

The sequential characterization of compactness is given by

Theorem 3.10. *(M, d) is compact $\iff$ every sequence in M has a convergent subsequence in M.*

Proof. By Lemma 3.5 and the definition of completenes:

$$\left\{ \begin{array}{c} \text{totally bounded} \\ + \\ \text{complete} \end{array} \right\} \iff \left\{ \begin{array}{c} \text{every sequence in } M \\ \text{has a Cauchy subsequence} \\ + \\ \text{Cauchy sequences converge} \end{array} \right\}$$

□

Compactness is an extremely useful property to have: if you happen to have a sequence in a compact space which does not converge, simply extract a convergent subsequence and use this one instead!

Corollary 3.11. *Let A be a subset of a metric space M. If A is compact, then A is closed in M (and totally bounded). If M is compact and A is closed, then A is compact.*

Proof. Assume that A is compact and let $x \in M$ and $(x_n)_n \subset A$ with $x_n \to x$. By Theorem 3.10, $(x_n)_n$ has a convergent subsequence whose limit is also in $A \Rightarrow x \in A$ so A is closed.
Assume M is compact, $A \subset M$ is closed. Given $(x_n)_n \subset A$, Theorem 3.10 supplies a convergent subsequence of $(x_n)_n$ which converges to a point $x \in M$. Since A is closed, we must have $x \in A$, so by Theorem 3.10 again, A is compact. □

Corollary 3.12. *Let (M, d) be compact and $f : M \to \mathbb{R}$ continuous. Then f attains its maximum and minimum, i.e., there are $x_{\min}, x_{\max} \in M$ such that*

$$f(x_{\min}) = \inf(f(x)|x \in M),$$
$$f(x_{\max}) = \sup(f(x)|x \in M),$$

In particular, inf *and* sup *are finite!*

Proof. Only for mininmum (otherwise look at $-f$).
Let $a := \inf(f(x)|x \in M)$. Note that there is always a minimizing sequence, i.e., a sequence $(x_n)_n \subset M$ such that

$$f(x_n) \to a \quad \text{as } n \to \infty.$$

Now if $(x_n)_n$ converges to some point $x \in M$, then we are done, since by continuity of f,

$$f(x) = \lim_{n\to\infty} f(x_n) = a = \inf(f(x)|x \in M).$$

If $(x_n)_n$ does not converge, use the fact that M is compact, so by Theorem 3.10 $(x_n)_n$ has a convergent subsequence and then use this subsequence instead! □

Corollary 3.13. *Let (N, ρ) be a metric space. If (M, d) is compact and $f : (M, d) \to (N, \rho)$ is continuous, then f is uniformly continuous.*

Proof. Recall the definition of uniform continuity:

$$\forall \varepsilon > 0 \exists \delta > 0 : x, y \in M, d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon.$$

So assume that f is not uniformly continuous. Then by negating the above one sees

$$\exists \varepsilon > 0 : \forall \delta > 0 \exists x, y \in M, d(x, y) < \delta \text{ and } \rho(f(x), f(y)) \geq \varepsilon.$$

Now fix this $\varepsilon > 0$ and let $\delta = \frac{1}{n}$. Then there must exist $x_n, y_n \in M, d(x_n, y_n) < \frac{1}{n}$ and $\rho(f(x_n), f(y_n)) \geq \varepsilon$. Since $(y_n)_n \subset M$ and M is compact, there exists a subsequence $(y_{n_l})_l$ of $(y_n)_n$ which converges to some point y. Look at $(x_{n_l})_l$. Again by compactness, there exists a subsequence $(x_{n_{l_k}})_k$ which converges to some point x. Since $x_{n_{l_k}} \to x$ and $y_{n_{l_k}} \to y$ we have

$$d(x, y) = \lim_{k\to\infty} d(x_{n_{l_k}}, y_{n_{l_k}}) = 0,$$

i,e, $x = y$.
But since $\rho(f(x_n), f(y_n)) \geq \varepsilon > 0$, we have

$$\lim_{k\to\infty} f(x_{n_{l_k}}) \neq \lim_{k\to\infty} f(y_{n_{l_k}})$$

so f is not continuous at x.
Thus f not uniformly continuous $\Rightarrow$ f not continuous $\iff$ f continuous $\Rightarrow$ f uniformly continuous. □