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Lectures Notes in Functinoal Analysis

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4 The sequence spaces $l^p(\mathbb{N}), 1 \leq p \leq \infty$

Definition 4.1. • *$l^\infty(\mathbb{N})$ is the space of all bounded sequences $x : \mathbb{N} \to \mathbb{F}$ equipped with the norm*

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

- *Let $1 \leq p < \infty$. $l^p(\mathbb{N})$ is the space of all sequences $x : \mathbb{N} \to \mathbb{F}$ for which $\sum\limits_{n\in\mathbb{N}} |x_n|^p < \infty$. With*

$$\|x\|_p := \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{\frac{1}{p}}$$

it becomes a normed vector space.

Lemma 4.2. *Let $1 \leq p \leq \infty$. Then $(l^p(\mathbb{N}), \|\cdot\|_p)$ is a normed vector space.*

Proof. Case 1: $p = \infty$ should be immediate.
Case 2: $1 \leq p < \infty$ is more complicated. It is not even obvious why $(l^p, \|\cdot\|_p)$ is a vector space. If $x \in l^p$ and $\alpha \in \mathbb{F}$, then $\alpha x \in l^p$ is clear, but if $x, y \in l^p(\mathbb{N})$ why is $x + y \in l^p(\mathbb{N})$?
Let $x, y \in l^p(\mathbb{N})$, i.e., $\|x\|_p, \|y\|_p < \infty$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n + y_n|^p &\leq \sum_{n=1}^{\infty} (2\max(|x_n|, |y_n|))^p \\ &= 2^p \sum_{n=1}^{\infty} \max(|x_n|, |y_n|) \leq 2^p(\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p) < \infty\end{aligned}$$

so $x + y \in l^p(\mathbb{N})$.
To show that $\|\cdot\|_p$ is a norm, we only have to check the triangle-inequality and for this we need some more help.

Lemma 4.3 (Hölder inequality)**.** *Let $1 \leq p \leq \infty$ and define the* ***dual exponent*** *$q \in [1, \infty]$ by*

$$q = \begin{cases} \infty, & \text{if } p = 1, \\ 1, & \text{if } p = \infty, \\ \frac{p}{p-1} (\text{ i.e., } q \text{ is such that } \frac{1}{p} + \frac{1}{q} = 1), & \text{if } 1 < p < \infty. \end{cases}$$

Then if $x \in l^p(\mathbb{N}), y \in l^q(\mathbb{N})$ and if $x - y$ is defined by $(x \cdot y)_n := x_n \cdot y_n, n \in \mathbb{N}$, then $x \cdot y \in l^1(\mathbb{N})$ and

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q.$$

Armed with this, we can show that $\|\cdot\|_p$ is a norm for $1 \leq p \leq \infty$.

Let $x, y \in l^p(\mathbb{N})$, then

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p \leq \sum_{n=1}^{\infty} (|x_n| + |y_n|)^p \\
&= \sum_{n=1}^{\infty} (|x_n| + |y_n|)(|x_n| + |y_n|)^{p-1} \\
&= \sum_{n=1}^{\infty} |x_n|(|x_n| + |y_n|)^{p-1} + \sum_{n=1}^{\infty} |y_n|(|x_n| + |y_n|)^{p-1} =: (\star)
\end{aligned}$$

We know already that $x + y \in l^p$. Let q be the dual exponent to p. Then $\frac{p}{q} = p - 1$ (with the convention that $\frac{1}{\infty} = 0$). So, since $(|x_n| + |y_n|)_n \in l^p$, one has

$$(|x_n| + |y_n|)^{p-1} = (|x_n| + |y_n|)^{\frac{p}{q}} \in l^q.$$

So the Hölder inequality applies to $(\star)$ and

$$\begin{aligned}
\sum_{n=1}^{\infty} (|x_n| + |y_n|)^p &= \sum_{n=1}^{\infty} |x_n|(|x_n| + |y_n|)^{\frac{p}{q}} + \sum_{n=1}^{\infty} |y_n|(|x_n| + |y_n|)^{\frac{p}{q}} \\
&\leq \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} (|x_n| + |y_n|)^{\frac{p}{q}\cdot q}\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\sum_{n=1}^{\infty} |y_n|^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} (|x_n| + |y_n|)^{\frac{p}{q}\cdot q}\Big)^{\frac{1}{q}} \\
&= (\|x\|_p + \|y\|_p)\Big(\sum_{n=1}^{\infty} (|x_n| + |y_n|)^p\Big)^{\frac{1}{q}}
\end{aligned}$$

$$\Rightarrow \quad \underbrace{\Big(\sum_{n=1}^{\infty} (|x_n| + |y_n|)^p\Big)^{1-\frac{1}{q}}}_{\left(\sum_{n=1}^{\infty} (|x_n|+|y_n|)^p\right)^{\frac{1}{p}} = \|x+y\|_p} \leq \|x\|_p + \|y\|_p.$$

So $\|x + y\|_p \leq \|x\|_p + \|y\|_p$. □

It remains to prove Hölder inequality. For this we need

Lemma 4.4 (Young's inequality)**.** *Let $1 < p < \infty$. Then for all $a, b \geq 0$*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For some suitable function G, we want to have an inequality of the form

$$a \cdot b \leq G(a) + F(b) \quad \forall a, b \geq 0$$

for a suitable function F. How to guess F?
Certainly F given by

$$F(b) := \sup_{a>0}(ab - G(a)) \qquad (\star\star)$$

works, since then

$$G(a) + F(b) \geq G(a) + ab - G(a) = ab.$$

So we need to find the supremum in $(\star\star)$. If $G(a) = \frac{1}{p}a^p$ and $1 < p < \infty$, then $G(0) = 0$ and $\lim_{a\to\infty}(ab - G(a)) = -\infty$ so there will be a point a (depending on b) for which $a \mapsto ab - G(a)$ is maximal.
At this point the derivative

$$\frac{d}{da}(ab - G(a)) = b - G'(a) = b - a^{p-1}$$

must be zero $\Rightarrow a = b^{1/(p-1)}$.

$$\begin{aligned}\Rightarrow F(b) &= ab - \frac{1}{p}a^p = a(b - \frac{1}{p}a^{p-1}) \\ &= b^{\frac{1}{p-1}}(b - \frac{1}{p}b) = b^{\frac{p}{p-1}}\frac{p}{p-1} = \frac{1}{q}b^q\end{aligned}$$

with $q = \frac{p}{p-1}$. □

Proof of Lemma 4.3. Let $(x_n)_n \in l^p$ and $(y_n)_n \in l^q, 1 \leq p \leq \infty$, q dual exponent of p.
The cases $p = 1$ or $p = \infty$ are easy (do them!).
So let $1 < p < \infty$.
Step 1: Assume $\|x\|_p = 1 = \|y\|_q$. Then

$$\|x \cdot y\|_1 \leq 1.$$

Indeed,

$$\|x \cdot y\|_1 \leq \sum_{n=1}^{\infty} |x_n y_n| = \sum_{n=1}^{\infty} |x_n||y_n|$$

and by Lemma 4.4

$$\begin{aligned}&\leq \sum_{n=1}^{\infty}\left(\frac{1}{p}|x_n|^p + \frac{1}{q}|y_n|^q\right) \\ &= \frac{1}{p}\|x\|_p^p + \frac{1}{q}\|y\|_q^q \\ &= \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

Step 2: Assume $x \neq 0, y \neq 0$. Then

$$\frac{\|x \cdot y\|_1}{\|x\|_p\|y\|_q} = \|\frac{x}{\|x\|_p} \cdot \frac{y}{\|y\|_q}\|_1 = \|\tilde{x} \cdot \tilde{y}\|_1$$

with $\tilde{x} = \left(\frac{x_n}{\|x\|_p}\right)_n, \tilde{y} = \left(\frac{y_n}{\|y\|_q}\right)_n$.
Note $\|\tilde{x}\|_p = 1 = \|\tilde{y}\|_q$. So by Step 1

$$\|\tilde{x} \cdot \tilde{y}\|_1 \leq 1,$$

hence

$$\|x \cdot y\|_1 = \|x\|_p \|y\|_q \|\tilde{x} \cdot \tilde{y}\|_1 \leq \|x\|_p \|y\|_q.$$

Theorem 4.5. *The spaces $(l^p(\mathbb{N}), \|\cdot\|_p)$ are Banach spaces, i.e., they are co plete.*

Proof. Only completeness remains: we do only $1 \leq p < \infty$.
We write $x = (x(j))_{j \in \mathbb{N}} \in l^p(\mathbb{N})$.
So let $(x_n)_n \subset l^p(\mathbb{N})$ be Cauchy.
Step 1: A candidate for the limit: Fix $j \in \mathbb{N}$ and consider

$$|x_n(j) - x_m(j)| \leq \Big(\sum_{l=1}^{\infty} |x_n(l) - x_m(l)|^p \Big)^{\frac{1}{p}} = \|x_n - x_m\|_p$$

$\Rightarrow (x_n(j))_n \subset \mathbb{F}$ is Cauchy. By completeness of $\mathbb{F}$ $x(j) := \lim_{n \to \infty} x_n(j)$ exis
Step 2: $x \in l^p(\mathbb{N})$!
Idea:

$$\|x\|_p^p = \sum_{j=1}^{\infty} |x(j)|^p = \sum_{j=1}^{\infty} \lim_{n \to \infty} |x_n(j)|^p \text{ “ = “ } \lim_{n \to \infty} \underbrace{\sum_{j=1}^{\infty} |x_n(j)|^p}_{\|x_n\|_p^p} < \infty.$$

Let $L \in \mathbb{N}$. Note that

$$\begin{aligned} \sum_{j=1}^{L} |x(j)|^p &= \sum_{j=1}^{L} \lim_{n \to \infty} |x_n(j)|^p \\ &= \liminf_{n \to \infty} \sum_{j=1}^{L} |x_n(j)|^p \\ &\leq \liminf_{n \to \infty} \|x_n\|_p^p < \infty \end{aligned}$$

Thus, using the monotone convergence theorem, we conclude that

$$\sum_{j=1}^{\infty} |x(j)|^p = \lim_{L \to \infty} \sum_{j=1}^{L} |x(j)|^p \leq (\liminf \|x_n\|_p)^p$$

$$\Rightarrow \|x\|_p \leq \liminf \|x_n\|_p,$$

so $x \in l^p$!
Step 3: $x_n \to x$ in l^p: Given $\varepsilon > 0$, there exists

$$N \in \mathbb{N} : \|x_n - x_m\|_p < \varepsilon \quad \forall n, m \geq N.$$

Let $L \in \mathbb{N}$.

$$\begin{aligned}\sum_{j=1}^{L} |x(j) - x_n(j)|^p &= \lim_{m\to\infty} \sum_{j=1}^{L} |x_m(j) - x_n(j)|^p \\ &\leq \limsup_{m\to\infty} \|x_m - x_n\|_p^p \\ &\leq \varepsilon^p \text{ for } n \text{ large enough}\end{aligned}$$

$\Rightarrow$ for $n \geq N$:

$$\|x - x_n\|_p^p = \lim_{L\to\infty} \sum_{j=1}^{L} |x(j) - x_n(j)|^p \leq \varepsilon^p$$

or $\|x - x_n\|_p \leq \varepsilon$ for all n large enough!

□

5 Hahn-Banach type theorems

5.1 Some preparations

Definition 5.1. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed vector spaces. A continuous linear map $T : X \to Y$ is called* ***operator****. If $Y = \mathbb{R}$ or $\mathbb{C}$ we call them functionals.*

Lemma 5.2. *Let X, Y be normed vector spaces and $T : X \to Y$ linear. Then the following are equivalent (t.f.a.e.):*

(a) T is continuous.

(b) T is continuous at 0.

(c) $\exists M \geq 0 : \|Tx\|_Y \leq M\|x\|_X \forall x \in X$.

(d) T is uniformly continuous.

Proof. $(c) \Rightarrow (d) \Rightarrow (a) \Rightarrow (b)$ is easy.
E.g.: $(c) \Rightarrow T$ is Lipschitz continuous, since

$$\|Tx - Tx_0\|_Y = \|T(x - x_0)\|_Y \leq M\|x - x_0\|_X.$$

So we only need to show $(b) \Rightarrow (c)$. Assume that (c) is wrong $\Rightarrow \forall n \in \mathbb{N} \exists x_n \in X : \|Tx_n\|_Y > n\|x_n\|_X \Rightarrow x_n \neq 0$. Then

$$y_n := \frac{x_n}{n\|x_n\|_X} \to 0 \text{ in } X.$$

But

$$\|Ty_n\|_Y = \frac{\|Tx_n\|_X}{n\|x_n\|_X} > 1$$

so $Ty_n \nrightarrow 0$, so T is not continuous at 0, a contradiction. □

Definition 5.3 (Operator-norm). *Given $T : X \to Y$ linear*

$$\|T\| := \|T\|_{X\to Y} := \inf(M \geq 0 | \|Tx\|_Y \leq M\|x\|_X \text{ for all } x \in X)$$

defines the ***operator-norm*** *of T.*

Note:

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X=1} \|Tx\|_Y = \sup_{\|x\|_X\leq 1} \|Tx\|_Y.$$

and

$$\|Tx\|_Y \leq \|T\|\|x\| \quad \forall x \in X. \tag{I.1}$$

Indeed, let $M_0 := \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$

$$\|Tx\|_Y = \frac{\|Tx\|_Y}{\|x\|_X}\|x\|_X \leq M_0\|x\|_X$$

$$\Rightarrow \|T\| \leq M_0.$$

On the other hand: given $\varepsilon > 0 \exists x_\varepsilon \neq 0$:

$$\|Tx_\varepsilon\|_Y \geq M_0(1-\varepsilon)\|x_\varepsilon\|_X$$

$$\Rightarrow \|T\| \geq M_0(1-\varepsilon) \quad \forall \varepsilon < 0$$

$$\Rightarrow \|T\| \geq M_0$$

and thus $\|T\| = M_0$, so (I.1) holds.

Definition 5.4. *Let X, Y be normed spaces.*

$$L(X,Y) := \{T : X \to Y | T \text{ is linear and continuous}\}$$

is again a vector space.

$$(S+T)(x) := Sx + Tx,$$
$$(\lambda T)(x) := \lambda Tx.$$

Proposition 5.5. *(a) $\|T\| = \sup_{\|x\|_X\leq 1} \|Tx\|_Y$ defines a norm on $L(X,Y)$.*

(b) If Y is complete, then $L(X,Y)$ is also complete.

Proof. (a) Looking closely reveals

$$\|\lambda T\| = |\lambda|\|T\|$$

$$\|T\| = 0 \Rightarrow T = 0 \text{ (the zero linear map)}.$$

Triangle-inequality:

$$\begin{aligned}\|S+T\| &= \sup_{\|x\|_X\leq 1} \underbrace{\|(S+T)x\|_Y}_{=\|Sx+Tx\|_Y} \\ &\leq \sup_{\|x\|_X\leq 1} \|Sx\|_Y + \sup_{\|x\|_X\leq 1} \|Tx\|_Y \\ &= \|S\| + \|T\|.\end{aligned}$$

(b) Let $(T_n)_n \subset L(X,Y)$ be Cauchy $\Rightarrow$ for fixed $x \in X$ $(T_n x)_n$ is Cauchy in Y!

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\| \|x\|_X.$$

By completeness of $Y \Rightarrow Tx := \lim_{n\to\infty} T_n x$ exists.
Step 1: T is linear. Indeed

$$\begin{aligned} T(\lambda x_1 + \mu x_2) &= \lim_{n\to\infty} \underbrace{T_n(\lambda x_1 + \mu x_2)}_{\lambda T_n x_1 + \mu T_n x_2} \\ &= \lambda \lim_{n\to\infty} T_n x_1 + \mu \lim_{n\to\infty} T_n x_2 \\ &= \lambda T x_1 + \mu T x_2. \end{aligned}$$

Step 2: $T \in L(X,Y)$, i.e., $\|T\| < \infty$ and $\|T - T_n\| \to 0$.
Indeed, let $\varepsilon > 0$ and choose $N_1 \in \mathbb{N}$ so that

$$\|T_n - T_m\| < \varepsilon \quad \forall n, m \geq N_1.$$

Let $x \in X$, $\|x\|_X \leq 1$. Choose $N_\varepsilon := N_\varepsilon(\varepsilon, x) \geq N_1$ so that

$$\|T_{N_\varepsilon} x - Tx\|_Y \leq \varepsilon$$

Thus, for every $x \in X$ with $\|x\|_X \leq 1$:

$$\begin{aligned} \|T_n x - Tx\|_Y &\leq \underbrace{\|T_n x - T_{N_\varepsilon} x\|_Y}_{=\|(T_n - T_{N_\varepsilon})x\|_Y \leq \|T_n - T_{N_\varepsilon}\| \|x\|_X \leq \|T_n - T_{N_\varepsilon}\|} + \underbrace{\|T_{N_\varepsilon} x - Tx\|_Y}_{\leq \varepsilon} \\ &\leq \underbrace{\|T_n - T_{N_\varepsilon}\|}_{\leq \varepsilon, n \geq N_1} + \varepsilon \\ &\leq 2\varepsilon \text{ for all } n \geq N_1 \end{aligned}$$

$$\begin{aligned} &\|Tx\|_Y \leq \|T_n x - Tx\|_Y + \|T_n x\|_Y \leq 2\varepsilon + \|T_n\| < \infty \\ &\|T - T_n\| = \sup_{\|x\|_X \leq 1} \|Tx - T_n x\| \leq 2\varepsilon \text{ for all } n \geq N_1, \end{aligned}$$

so $T_n \to T$ in operator norm.

□

Definition 5.6. *Given a normed vector space X, its **dual** space is the space $X' = X^* := L(X, \mathbb{F})$ of continuous linear functionals.*

Corollary 5.7. *For any normed vector space X, its dual X' equipped with the norm*

$$\|x'\|_{X'} := \sup_{\|x\|_X \leq 1} |x'(x)| = \sup_{\|x\|_X = 1} |x'(x)|$$

is a Banach space.

5.2 The analytic form of Hahn-Banach: extension of linear functionals

Definition 5.8. *Let E be a vector space. A map $p : E \to \mathbb{R}$ is* ***sub-linear*** *if*

(a) $p(\lambda x) = \lambda p(x), \forall \lambda \geq 0, \forall x \in E.$

(b) $p(x+y) \leq p(x) + p(y), \forall x, y \in E.$

Example 5.9. *(i) Every semi-norm is sub-linear.*

(ii) Every linear functional on a real vector space is sub-linear.

(iii) On $l^\infty(\mathbb{N}, \mathbb{R})$ = bounded real-valued sequences, $t = (t_n)_n \mapsto \limsup_{n\to\infty} t_n$ is sub-linear.
On $l^\infty(\mathbb{N}, \mathbb{C})$, $t = (t_n)_n \mapsto \limsup_{n\to\infty} Re(t_n)$ is sub-linear.

(iv) A sub-linear map is often called Minkowski functional.

Theorem 5.10 (Hahn-Banach, analytic form). *Let E be a real vector space, $p : E \to \mathbb{R}$ sub-linear, $G \subset E$ a subspace, and $g : G \to \mathbb{R}$ a linear functional with*

$$g(x) \leq p(x) \qquad \forall x \in G.$$

Then there exists a linear functional $f : E \to \mathbb{R}$ which extends g, i.e., $g(x) = f(x)\ \forall x \in G$, such that

$$f(x) \leq p(x) \qquad \forall x \in E.$$

For the proof we need Zorn's lemma, which is an important property of ordered sets.

Some notations:

- Let P be a set with a partial order relation $\leq$. A subset $Q \subset P$ is **totally ordered** if for any $a, b \in Q$ either $a \leq b$ or $b \leq a$ (or both!) holds.
- Let $Q \subset P$, then $c \in P$ is an **upper bound** for Q if $a \leq c$ for all $a \in Q$.
- We say that $m \in P$ is a **maximal element** of P if there is no element $x \in P$ such that $m \leq x$ except for $x = m$.
  Note that a maximal element of P need not be an upper bound for P!
- We say that P is **inductive** if every totally ordered subset $Q \subset P$ has an upper bound.

Lemma 5.11 (Zorn). *Every non-empty ordered set which is inductive has a maximal element.*

Proof of Theorem 5.10. We say that h extends g if

$$D(h) \supset D(g) \quad \text{and} \quad h(x) = g(x)\ \forall x \in D(g).$$

Consider the set

$$P = \left\{ \begin{array}{lr} h : E \supset D(h) \to \mathbb{R} | & D(h) \text{ is a linear subspace of } E, \\ & h \text{ is linear}, G \subset D(h), h \text{ extends } g, \\ & h(x) \leq p(x) \quad \forall x \in D(h) \end{array} \right\}.$$

Note: $P \neq \emptyset$ since $g \in P$!
On P we define the order $h_1 \leq h_2 \iff h_2$ extends h_1.
Step 1: P is inductive.
Indeed, let $Q \subset P$ be totally ordered. Write $Q = (h_i)_{i \in I}$ and set

$$D(h) := \bigcup_{i \in I} D(h_i), \quad h(x) := h_i(x) \text{ if } x \in D(h_i) \text{ for some } i \in I.$$

It is easy to see that this definition is consistent and that h is an upper bound for Q.
Step 2: By Step 1 and Zorn's lemma, P has a maximal element $f \in P$.
Claim: $D(f) = E$ (which finishes the proof).
Assume that $D(f) \neq E$. Let $x_0 \notin D(f)$ and set $D(h) := D(f) + \mathbb{R}x_0$ and for $x \in D(f)$ set

$$h(x + tx_0) := f(x) + t\alpha, \quad t \in \mathbb{R},$$

where we will choose α so that $h \in P$. For this we need

$$f(x) + t\alpha \leq p(x + tx_0). \tag{I.2}$$

Let $t > 0$. Then

$$\begin{aligned} \text{(I.2)} \iff t\alpha &\leq p(x + tx_0) - f(x) \\ \iff \alpha &\leq \frac{1}{t}p(x + tx_0) - \frac{1}{t}f(x) \\ &= p(\frac{x}{t} + x_0) - f(\frac{x}{t}) \\ &= p(u + x_0) - f(u), \end{aligned}$$

where $u := \frac{x}{t}$.
If $t < 0$, then

$$\begin{aligned} \text{(I.2)} \iff t\alpha &\leq p(x + tx_0) - f(x) \\ \iff -\alpha &\leq \frac{1}{-t}p(x + tx_0) - \frac{1}{-t}f(x) \\ &= p(\frac{x}{-t} + x_0) - f(\frac{x}{-t}) \\ &= p(w + x_0) - f(w) \\ \iff \alpha &\geq f(w) - p(w - x_0), \end{aligned}$$

where $w := \frac{x}{-t}$.
Thus (I.2) holds if

$$f(w) - p(w - x_0) \leq \alpha \leq p(u + x_0) - f(u) \quad \forall u, w \in D(f). \tag{I.3}$$

Since $f \in P$, we have

$$f(x) \leq p(x) \quad \forall x \in D(f).$$

Hence $\forall u, w \in D(f)$ it holds

$$\begin{aligned} f(u) + f(w) &= f(u + w) \\ &\leq p(u + w) \\ &= p(u + x_0 + w - x_0) \\ &\leq p(u + x_0) + p(w - x_0) \end{aligned}$$