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كلية التربية للعلوم الصرفة

قسم الرياضيات / ماجستير

مقرر: التحليل الدالي

المحاضرة الاولى (5)

(المصدر)

Lectures Notes in Functinoal Analysis WS 2012 – 2013

- (1) $0 < \varepsilon_n < \frac{1}{2}\varepsilon_{n-1}$, in particular $\varepsilon_n < 2^{-n}\varepsilon$
- (2) $B_{\overline{\varepsilon_n}}(x_n) \subset B_{2\varepsilon_n}(x_n) \subset \mathcal{O}_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \subset \cdots \subset \mathcal{O}_1 \cap \mathcal{O}_2 \ldots \mathcal{O}_n \cap B_{\varepsilon}(x_0)$ for all $n \in \mathbb{N}$. In particular, $x_n \subset B_{\varepsilon_N}(x_N) \subset B_{2^{-N}\varepsilon}(x_0)$ for all $n \geq N$.
- $\Rightarrow (x_n)_n$ is Cauchy, M is complete, so $x = \lim x_n \in M$ exists

$$\Rightarrow x \in B_{\overline{\varepsilon_N}}(x_N) \quad \forall N \in \mathbb{N}.$$

So
$$D \cap B_{\varepsilon}(x_0) \neq \emptyset \ \forall \varepsilon > 0, x_0 \in M$$
, so D is dense in M.

Corollary 6.4 (Baire). Let M be a complete metric space, $(F_n)_n \subset M$ closed such that $M = \bigcup_{n \in \mathbb{N}} F_n$. Then there exists $n_0 \in \mathbb{N} : int F_{n_0} \neq \emptyset$. So a complete metric space is not meager.

Proof. F_n is closed. If $intF_n = \emptyset$, by Theorem 6.2(a), $F = \bigcup_{n \in \mathbb{N}} F_n$ has empty interior, so M = F has empty interior, but $M^o = int(M) = M$, a contradiction.

6.2 Application I: The set of discontinuities of a limit of continuous functions

Theorem 6.5. Assume that $(f_n)_n : M \to \mathbb{C}$ are continuous, M is complete metric space and

$$f(x) := \lim_{n \to \infty} f_n(x)$$

exists for all $x \in M$. Then the set of points where f is discontinuous is (at most) meager. In other words, the set of points where f is continuous is the complement of a meager set, in particular it is dense.

Proof. Let D= set of discontinuities of f. The oscillations of the function f at a point x are

$$osc(f)(x) := \lim_{r \to 0} w(f)(r, x) = \inf_{r > 0} w(f)(r, x)$$

with $w(f)(r,x) := \sup_{y,z \in B_r(x)} |f(y) - f(z)|$ (which is decreasing in r). So $osc(f)(x) < \varepsilon \iff \exists$ ball B centered at x with $|f(y) - f(z)| < \varepsilon \ \forall z, y \in B$. Note also

$$osc(f)(x) = 0 \Leftrightarrow f \text{ is continuous at } x$$
 (I.9)

$$\forall \varepsilon > 0 \quad E_{\varepsilon} := \{ x \in M | osc(f)(x) < \varepsilon \} \text{ is open}$$
 (I.10)

(I.9) is immediate and for (I.10) note that if $x \in E_{\varepsilon}$ there exists r > 0 with

$$\sup_{y,z\in B_r(x)}|f(y)-f(z)|<\varepsilon.$$

So if $\tilde{x} \in B_{\frac{\varepsilon}{2}}(x)$ then $\tilde{x} \in E_{\varepsilon}$ since $B_{\frac{\varepsilon}{2}}(\tilde{x}) \subset B_{\varepsilon}(x)$ and hence

$$\sup_{y,z\in B_{\frac{\varepsilon}{4}}(\tilde{x})}|f(y)-f(z)| \le \sup_{y,z\in B_{\varepsilon}(x)}|f(y)-f(z)| < \varepsilon.$$

Thus $B_{\frac{\varepsilon}{2}}(\tilde{x}) \subset E_{\varepsilon}$, so E_{ε} is open.

We need one more

Lemma 6.6. Let $(f_n)_n$ be a sequence of continuous functions on a complete metric space M and $f_n(x) \to f(x) \ \forall x \in M$. Then given any open ball $B \subset M$ and $\varepsilon > 0$, there exist an open ball $B_0 \subset B$ and $m \in \mathbb{N}$ such that $|f_m(x) - f(x)| \le \varepsilon \ \forall x \in B_0$.

Proof. Let Y be a closed ball in M and recall that Y is itself a complete metric space. Define

$$E_{l} := \{ x \in Y | \sup_{j,k \geq l} |f_{j}(x) - f_{k}(x)| \leq \varepsilon \}$$

$$= \bigcap_{j,k \geq l} \underbrace{\{ x \in Y | |f_{j}(x) - f_{k}(x)| \leq \varepsilon \}}_{\text{closed since } f_{n} \text{ is continuous}}.$$

So E_l is closed and since $f_n(x) \to f(x) \ \forall x \in M$ we have $Y = \bigcup_{l=1}^{\infty} E_l$.

By Corollary 6.4 applied to Y, some set, say E_m , must contain an open ball B_0 . But then

$$\sup_{j,k \ge l} |f_j(x) - f_k(x)| \le \varepsilon \quad \forall x \in B_0$$

and letting $k \to \infty$ one sees

$$|f_m(x) - f_k(x)| \le \varepsilon \quad \forall x \in B_0.$$

To finish the proof of Theorem 6.5 define

$$F_n := \{ x \in M | osc(f)(x) \ge \frac{1}{n} \}.$$

So $F_n = E_{\frac{1}{n}}^c$ (from (I.10)) so F_n is closed and $D = \bigcup_{n \in \mathbb{N}} F_n$ is the set of discontinuities of f.

Final claim: Each F_n is nowhere dense!

Indeed, if not, let B be open ball with $B \subset F_n$. Then setting $\varepsilon = \frac{1}{4n}$ in Lemma 6.6, we get an open ball $B_0 \subset B$ and $m \in \mathbb{N}$ such that

$$|f_m(x) - f(x)| \le \frac{1}{4n} \quad \forall x \in B_0.$$

 f_m is continuous $\Rightarrow \exists$ ball $B' \subset B_0$ such that

$$|f_m(y) - f_m(z)| \le \frac{1}{4n} \quad \forall y, z \in B'.$$

Then

$$|f(y) - f(z)| \le |f(y) - f_m(y)| + |f_m(y) - f_m(z)| + |f_m(z) - f(z)|$$

$$\le \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4n} = \frac{3}{4n} < \frac{1}{n} \quad \forall y, z \in B' \subset B \subset F_n.$$

So if x' is the center of B' then

$$osc(f)(x') < \frac{1}{n},$$

which contradicts $x' \in F_n!$

6.3 Application II: Continuous but nowhere differentiable functions

Consider the complete metric space C([0,1]) with norm $||f||_{\infty} := \sup_{x \in [0,1]} |f(x)|$ and metric $d(f,g) = ||f-g||_{\infty}$.

Theorem 6.7. The set of functions in C([0,1]) which are nowhere differentiable is generic (in particular, it is dense!).

Proof. Let D = set of functions $f \in C([0,1])$ which are differentiable at at least one point. We have

$$D \subset \bigcup_{N \in \mathbb{N}} \underbrace{\{f \in C([0,1]) | \exists x^* \in [0,1] : \forall x \in [0,1] | f(x) - f(x^*)| \le N|x - x^*|\}}_{=:E_N}$$
(I.11)

Claim:

- (a) E_N is closed.
- (b) E_N is nowhere dense, i.e., it has empty interior.

Then Theorem 6.2 yields the claim.

Proof of (a). Let $\{f_n\} \subset E_N$ with $f_n \to f$. Let x_n^* be the point for which (I.11) holds with f replaced by f_n . [0,1] is compact $\Rightarrow \exists (x_{n_k}^*)$ which converges to a limit $x^* \in [0,1]$. Then

$$|f(x) - f(x^*)| \le |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_{n_k}(x^*)| + |f_{n_k}(x^*) - f(x^*)|.$$
(I.12)

Since $||f_n - f||_{\infty} \to 0$, for $\varepsilon > 0 \; \exists K$ such that

$$\forall k > K \quad |f(x) - f_{n_k}(x)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f_{n_k}(x^*) - f(x^*)| < \frac{\varepsilon}{2}.$$

For the middle term in (I.12) note that $f_{n_k} \in E_N$ so

$$|f_{n_k}(x) - f_{n_k}(x^*)| \le |f_{n_k}(x) - f_{n_k}(x_{n_k}^*)| + |f_{n_k}(x_{n_k}^*) - f_{n_k}(x^*)|$$

$$\le N|x - x_{n_k}^*| + N|x_{n_k}^* - x^*|$$

and so

$$|f(x) - f(x^*)| \le \varepsilon + N|x - x_{n_k}^*| + N|x_{n_k}^* - x^*|$$
$$\to \varepsilon + |x - x^*| + N \cdot 0 \quad \text{as } k \to \infty.$$

Proof of (b). Let $P \subset C([0,1])$ be the subspace of all continuous piecewise linear functions. For 0 < M let $P_M \subset P$ be the set of continuous piecewise linear functions with slope $\geq M$ or $\leq -M$. Think of P_M as the set of "zig-zag" functions!

Key fact: $P_M \cap E_N = \emptyset$ if M > N!

Lemma 6.8. $\forall M > 0$ P_M is dense in C([0,1]).

Now we finish the proof that E_N has no interior points: Let $f \in E_N$ and $\varepsilon > 0$. Fix M > N, then $\exists h \in P_M$ with $||f - h|| < \varepsilon$ and $h \notin E_N$ since $P_M \cap E_N = \emptyset$ when M > N. So no open ball around f is entirely contained in E_n , i.e., E_N has no interior.

Proof of Lemma 6.8. Step 1: P is dense in C([0,1]): Let $f \in C([0,1])$. Then f is uniformly continuous, since [0,1] is compact. So there exists $g \in P$ with $||f-g|| < \varepsilon$. Indeed, since f is uniformly continuous $\exists \delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \forall |x - y| < \delta.$$

Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ and let g be the piecewise linear function on each interval $\left[\frac{k}{n}, \frac{k+1}{n}\right], k = 0, \ldots, n-1$ with $g(\frac{k}{n}) := f(\frac{k}{n}), \ g(\frac{k+1}{n}) := f(\frac{k+1}{n})$ and linearly interpolated in between. Then $||f - g|| < \varepsilon!$ Step 2: P_M is dense in P: Let g(x) = ax + b for $0 \le x \le \frac{1}{n}$ and

$$\varphi_{\varepsilon}(x) = g(x) + \varepsilon, \quad \psi_{\varepsilon}(x) = g(x) - \varepsilon.$$

Begin at g(0), travel a slope +M until you intersect φ_{ε} . Reverse direction and travel on a line segment of slope -M until you intersect ψ_{ε} . This yields a function $h \in P_M$ with

$$\psi_{\varepsilon}(x) \le h(x) \le \varphi_{\varepsilon}(x) \quad \forall 0 \le x \le \frac{1}{n}$$

SO

$$|g(x) - h(x)| \le \varepsilon$$
 in $[0, \frac{1}{n}]$.

Then begin at $h(\frac{1}{n})$ and repeat the argument on the interval $[\frac{1}{n}, \frac{2}{n}]$ and continue in this fashion.

$$\Rightarrow$$
 get a function $h \in P_M$ with $||g - h|| \le \varepsilon$. So $||f - h|| \le 2\varepsilon$

6.4 Application III: The uniform boundedness principle

Recall: Let E, F be normed vector spaces. $L(E, F) = \text{vector space of all bounded linear operators } T : E \to F \text{ with the norm } ||T|| := \sup_{x \in E, ||x||_E \le 1} ||Tx||.$

Theorem 6.9 (Banach-Steinhaus uniform boundedness principle). Let E be a Banach space and F be a normed vector space, $(T_i)_{i\in I}$ be a family (not necessarily countable) of continuous linear operators, $T_i \in L(E, F) \ \forall i \in I$. Assume that

$$\sup_{i \in I} ||T_i x|| < \infty \ \forall x \in E. \tag{I.13}$$

Then

$$\sup_{i \in I} ||T_i|| < \infty, \tag{I.14}$$

i.e., $\exists C < \infty : ||T_i x|| \le C||x|| \ \forall x \in E, \forall i \in I.$

Remark 6.10. The conclusion of Theorem 6.9 is quite remarkable and surprising. Just having the pointwise estimate $\sup_{i \in I} ||T_i x||$ we get $\sup_{i \in I} \sup_{\|x\| \le 1} ||T_i x|| \infty$.

Proof. $\forall n \in \mathbb{N} \text{ let}$

$$F_n := \{ x \in E | \forall i \in I, ||T_i x|| \le n \}.$$

 F_n is closed and $\bigcup_{n\in\mathbb{N}} F_n = E$. By Corollary $6.4 \Rightarrow \exists m \in \mathbb{N} : int F_m \neq \emptyset$. Then $\exists x_0 \in F_m, r > 0, B_r(x_0) \subset F_m$. Then

$$||T_i(x_0 + r \cdot z)|| \le m, \quad ||z|| \le 1$$

$$\Rightarrow ||T_{i}(z)|| = \frac{1}{r}||T_{i}(rz)|| = \frac{1}{r}||T_{i}(x_{0} + rz) - T_{i}(x_{0})||$$

$$\leq \frac{1}{r}\underbrace{||T_{i}(x_{0} + rz)||}_{\leq m} + \frac{1}{r}\underbrace{||T_{i}(x_{0})||}_{\leq m} \leq \frac{2m}{r} \quad \forall z \in E, ||z|| \leq 1.$$

Corollary 6.11. Let E, F be Banach spaces, $(T_n)_n \subset L(E, F)$ such that for $\forall x \in E, T_n x$ converges and let $Tx : \lim_{n \to \infty} T_n x$. Then

- (a) $\sup_{n\in\mathbb{N}} ||T_n|| < \infty$.
- (b) $T \in L(E, F)$.
- (c) $||T|| \le \liminf_{n\to\infty} ||T_n||$.

Proof. (a) Follows from Theorem 6.9 immediately.

- (b) Also follows from Theorem 6.9 immediately.
- (c)

$$||Tx|| \leftarrow ||T_n x|| \le C||x|| \ \forall x \in E$$

$$||Tx|| \leftarrow ||T_nx|| \le ||T_n|| ||x||$$

$$||T|| \leq \liminf ||T_n||$$

Corollary 6.12. Let $B \subset G$ and G be a normed vector space (not necessarily complete). Then the following are equivalent

- (a) B is bounded.
- (b) f(B) is bounded for $\forall f \in G^*$.

Proof. $(a) \Rightarrow (b)$ is obvious.

 $(b) \Rightarrow (a)$: Recall that G^* is a Banach space.

For $x \in B$ and $f \in G^*$ let $T_x(f) := f(x)$, $T_x(f)$ is linear and bounded, because

$$\sup_{f \in G^*, ||f|| = 1} |T_x(f)| \le ||x|| ||f|| \le ||x||.$$

By Theorem 6.9 and (b) with $E = G^*, F = \mathbb{F}$ and I = B, we conclude

$$||T_x(f)|| \le C||f|| \quad \forall x \in B, \forall f \in G^*.$$

Then for $\forall x \in B$

$$||x|| = \sup_{f \in G^*, ||f|| \le 1} |f(x)| = \sup_{f \in G^*, ||f|| \le 1} |T_x(f)| \le C||x||.$$

Notation: $(x_n)_n \subset E$ converges weakly to $x \in E$ $(x_n \rightharpoonup x)$ if $\forall f \in E^*$ it holds $f(x_n) \to f(x)$.

Corollary 6.13. Weakly convergent sequences are bounded.

Proof. If $(x_n)_n$ converges weakly, then for any $f \in E^*, (f(x_n))_n$ is bounded. The result follows from Corollary 6.12.

Corollary 6.14 (Statement dual to 6.12). Let G be a Banach space and $B^* \subset G^*$. Then the following are equivalent

- (a) $\forall x \in G$ the set $B^*(x) := \{f(x) | f \in B^*\}$ is bounded.
- (b) B* is bounded.

Proof. (b) \Rightarrow (a) is obvious $(\exists M, ||f|| \leq M \ \forall f \in B^*)$. (a) \Rightarrow (b): We apply Theorem 6.9 with $E = G, F = \mathbb{F}, I = B^*$. For every $f \in B^*$ we set $T_f(x) := f(x), \ x \in G$. Due to (a) and Theorem 6.9 $\exists C < \infty$

$$|f(x)| = |T_f(x)| \le C||x|| \quad \forall f \in B^*, x \in G.$$

By definition

$$||f|| = \sup_{x \in F, ||x|| \le 1} |f(x)| \le C \quad \forall f \in B^*,$$

i.e., B^* is bounded.

6.5 Application IV: The Open Mapping and the Closed Graph theorems

Theorem 6.15 (Open Mapping). Let E, F be Banach spaces and $T \in L(E, F)$ be surjective. Then there exists C > 0 such that

$$T(B_1^E(0)) \supset B_c^F(0).$$
 (I.15)

Remark 6.16. Property (I.15) ensures that the image under T of any open set in E is open in F.

Indeed, let U be open in E. Fix $y_0 \in T(U)$ so $y_0 = Tx_0, x_0 \in U$. Let $r_0 > 0$ be such that $B_{r_0}(x_0) \subset U$. Due to Theorem 6.15 it holds $T(B_{r_0}(0)) \supset B_{cr_0}(0)$ (use linearity).

$$B_{cr_0}(y_0) = y_0 + B_{cr_0}(0) \subset T(x_0) + T(B_{r_0}(0))$$
$$= T(x_0 + B_{r_0}(0)) = T(\underbrace{B_{r_0}(x_0)}_{\subset U})$$

 $\Rightarrow T(U)$ is open.

Corollary 6.17. Let E, F be Banach spaces, $T \in L(E, F)$ bijective (i.e., injective and surjective). Then $T^{-1} \in L(F, E)$.

Proof. Obviously, T^{-1} exists and it is linear. By (I.15)

$$T^{-1}T(B_1^E(0)) \supset T^{-1}B_C^F(0)$$

$$\Rightarrow B_1^E(0) \supset T^{-1}B_c(0).$$

So if $y \in F, ||y|| < C \Rightarrow ||T^{-1}(y)|| < 1$

$$\Rightarrow \|T^{-1}y\|<\frac{1}{C},\quad \|y\|\leq 1$$

$$||T^{-1}|| \le 1.$$

Corollary 6.18. Let E be a vector space with two norms $\|\cdot\|_1, \|\cdot\|_2$ and assume that E is complete w.r.t. either norm and there exists C > 0 such that $\|x\|_2 \leq C\|x\|_1 \ \forall x \in E$. Then the two norms are equivalent, i.e., there exists $C_1 > 0$ such that $\|x\|_1 \leq C_1 \|x\|_2 \ \forall x \in E$.

Proof. Apply Corollary 6.17 with $E = (E, \|\cdot\|_1), F = (E, \|\cdot\|_2), T = Id.$

Proof of Theorem 6.15. Step 1: Assume T is linear surjective operator from E onto F. Then there exists c > 0 such that

$$\overline{T(B_1(0))} \supset B_{2c}(0).$$
 (I.16)

Indeed, set

$$F_n := n \overline{T(B_1(0))}.$$

T is surjective $\Rightarrow F = \bigcup_{n=1}^{\infty} F_n$. So by Baire there exists $m \in \mathbb{N} : int(F_m) \neq \emptyset$. By linearity $int(\overline{T(B_1(0))}) \neq \emptyset$! Pick c > 0 and $y_0 \in F$ such that

$$B_{4c}(y_0) \subset \overline{T(B_1(0))},$$

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in particular

$$y_0 \in \overline{T(B_1(0))}. (I.17)$$

By symmetry

$$-y_0 \in \overline{T(B_1(0))}. (I.18)$$

Adding (I.17) and (I.18) we get

$$B_{4c}(0) \subset \overline{T(B_1(0))} + \overline{T(B_1(0))}$$

and since $\overline{T(B_1(0))}$ is convex,

$$\overline{T(B_1(0))} + \overline{T(B_1(0))} = 2\overline{T(B_1(0))}$$

so (I.16) holds.

Step 2: Assume $T \in L(E, F)$ and (I.16) holds. Then (I.15) holds, i.e. $\overline{T(B_1(0))} \supset \overline{B_c(0)}$. Indeed, choose any $y \in F$, ||y|| < c.

<u>Aim:</u> Find some $x \in E$ such that ||x|| < 1 and Tx = y (because then (I.15) holds! (why?)).

By (I.16) we know that

$$\forall \alpha > 0 \text{ and } \tilde{y} \in F \text{ with } ||\tilde{y}|| < \alpha C$$

$$\exists z \in E \text{ with } ||z|| < \frac{\alpha}{2} \text{ and } ||\tilde{y} - Tz|| < \varepsilon. \tag{I.19}$$

(Hint: Use (I.16) and linearity to see this) Choosing $\varepsilon = \frac{c}{2}$ we find $z_1 \in E$ such that

$$||z_1|| < \frac{1}{2}$$
 and $||y - Tz_1|| < \frac{1}{2}C$.

Now apply (I.19) to $\tilde{y} = y - Tz_1$. Since $\|\tilde{y}\| < \frac{1}{2}C$, $\alpha = \frac{1}{2}$ and by (I.19) with $\varepsilon = \varepsilon_2 = \frac{C}{2^2}$, $\exists z_2 \in E$ with

$$||z_2|| < \frac{1}{4}$$
 and $||\tilde{y} - Tz_2|| = ||y - Tz_1 - Tz_2|| < \frac{\alpha}{2}c = \frac{c}{4}$.

Proceeding inductively, using (I.19) repeatedly with $\varepsilon = \varepsilon_n = \frac{c}{2^n}$, $\alpha = \alpha_n = \frac{1}{2^n}$ we obtain a sequence $(z_n)_n$ such that

$$||z_n|| < \frac{1}{2^n}$$
 and $||y - T(z_1 + z_2 + \dots z_n)|| < \frac{C}{2^n}$ $\forall n \in \mathbb{N}$.

So $x_n := z_1 + \dots z_n$ is Cauchy and hence $x_n \to x$ for some $x \in E$. Clearly

$$||x|| \le \sum_{n=1}^{\infty} ||z_n|| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

and since T is continuous we have y = Tx.

Theorem 6.19 (Closed Graph). Let E, F be Banach spaces and T a linear operator from E to F. Then

T is continuous \iff The graph of T is closed.

Remark 6.20. • Assume that $T: E \to F$. The graph of T is the set $G(T) := \{(x, T(x)) | x \in E\} \subset E \times F$.

• The set $G(T) \subset E \times F$ is closed if for every sequence $(x_n)_n \subset E$ for which $x_n \to x$ and $y_n := Tx_n \to y$ we have y = Tx.

Proof. " \Rightarrow ": Clear by continuity of T! " \Leftarrow ": Consider the two norms on E:

$$||x||_1 := ||x||_E + ||Tx||_F$$
 and $||x||_2 := ||x||_E$.

The norm $\|\cdot\|_1$ is called the graph norm.

E is a Banach space w.r.t. $\|\cdot\|_2$ by assumption and certainly

$$||x||_2 < ||x||_1 \quad \forall x \in E.$$

Let $(x_n)_n \subset E$ be Cauchy w.r.t. $\|\cdot\|_2$, i.e., $\forall \varepsilon > 0 \exists N : \|x_n - x_m\|_2 < \varepsilon \ \forall n, m \ge N$. Then $y_n := Tx_n$ is Cauchy in F and x_n is Cauchy in E. Therefore $x = \lim x_n$ and $y = \lim Tx_n$ exist. Since G(T) is closed, it follows that y = Tx. Thus

$$||x - x_n||_1 = ||x - x_n||_E + ||y - Tx_n||_F \to 0$$
 as $n \to \infty$

so x_n converges to x also in $\|\cdot\|_1$ norm, i.e., $(E, \|\cdot\|_1)$ is complete! By Corollary 6.18 the two norms are equaivalent, i.e., there exists c>0 such that

$$||x||_1 \le c||x||_2 = c||x||_E$$

so

$$||Tx||_F \le ||x||_E + ||Tx||_F = ||x||_1 \le c||x||_E.$$

7 Weak Topologies. Reflexive Spaces. Separable Spaces. Uniform Convexity

7.1 The coarsest topology for which a collection of maps becomes continuous

<u>Recall:</u> Given a set X a topology τ on X is a collection of subsets of X, called the open sets, such that

- (1) $\emptyset \in \tau, X \in \tau$,
- (2) arbitrary unions of sets in τ are in τ ,
- (3) finite intersections of sets in τ are in τ .