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Lectures Notes in Functinoal Analysis

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(2) is called τ is stable under arbitrary unions (or $\bigcup_{\text{arbitrary}}$), (3) is called τ is stable under finite intersections (or $\bigcap_{\text{finite}}$).
A set X with a topology τ is called topological space.

Suppose X is a set (no structure yet) and $(Y_i)_{i\in I}$ a collection of topological spaces, and $(\varphi_i)_{i\in I}$ a collection of maps $\varphi_i : X \to Y_i$.

Problem 1: Construct a topology on X that makes all the maps $(\varphi_i)_{i\in I}$ continuous. Can one find a topology on X which is most economical in the sense that it contains the fewest open sets?

Note: If X is equipped with the discrete topology, then all subsets of X are open and hence every map $\varphi_i : X \to Y_i$ is continuous. But this topology is huge!

Want: The cheapest topology! It is called the coarsest or weakest topology associated with $(\varphi_i)_{i\in I}$,

If $\omega_i \subset Y_i$ is open then $\varphi_i^{-1}(\omega_i)$ is necessarily open in τ and as ω_i varies in the open subsets of Y_i and i runs through I, one gets a family of open subsets which is necessarily open in X! Call this $(U_\lambda)_{\lambda\in\Lambda}$.
More precisely: (Y_i, τ_i) topological spaces, $\varphi_i : X \to Y_i$,

$$\Lambda := \{\lambda = (i, \omega_i) | i \in I, \omega_i \in \tau_i\},$$

$$U_\lambda = \varphi_i^{-1}(\omega_i).$$

Catch: $(U_\lambda)_{\lambda\in\Lambda}$ does not need to be a topology!

$\Rightarrow$ Problem 2: Given a set X and a family $(U_\lambda)_{\lambda\in\Lambda}$ of subsets of X, construct the cheapest topology τ on X which contains $(U_\lambda)_{\lambda\in\Lambda}$.

So τ must be stable under $\bigcap_{\text{finite}}$ and $\bigcup_{\text{arbitrary}}$ and $U_\lambda \subset \tau\ \forall\lambda \in \Lambda$.

Step 1: Consider the enlarged family of all finite intersections of sets in $(U_\lambda)_{\lambda\in\Lambda}$: $\bigcap_{\lambda\in\Gamma} U_\lambda$, where $\Gamma \subset \Lambda$ is finite. Call this family Φ. It is stable under $\bigcap_{\text{finite}}$.

Step 2: Φ need not be stable under $\bigcup_{\text{arbitrary}}$ $\Rightarrow$ consider families $\mathcal{F}$ obtained from Φ by taking arbitrary unions of sets in Φ. So $\mathcal{F}$ is stable under $\bigcup_{\text{arbitrary}}$.

Lemma 7.1. *$\tau := \bigcup_{arbitrary} \bigcap_{finite} U_\lambda$ is stable under $\bigcap_{finite}$. Hence τ is a topology!*

Proof. See any book on point set topology. □

A **basis of a neighborhood** of a point $x \in X$ is a family $(U_i)_{i\in I}$ of open sets containing x, such that any open set containing x contains an open set from the basis (i.e., from $(U_i)_{i\in I}$).

Example: In a metric space X, take the open balls centered at $x \in X$.

In our situation: Given $x \in X$, V_i a neighborhood of $\varphi_i(x)$ in Y_i

$$\bigcap_{\text{finite}} \varphi_i^{-1}(V_i)$$

yields a basis of neighborhoods of x in X.

In the following, we equip X with the topology τ which is the weakest (smallest, coarsest) topology for which all the $\varphi_i : X \to Y_i$ are continuous for all $i \in I$.

Proposition 7.2. *Let $(x_n)_n \subset X$. Then $x_n \to x$ in τ (i.e., for any $U \in \tau, x \in U, x_n \in U$ for almost all n) $\iff$ $\varphi_i(x_n) \to \varphi_i(x)$ as $n \to \infty$ $\forall i \in I$.*

Proof. $\Rightarrow$: Simple, since by definition φ_i is continuous $\forall i \in I$.
$\Leftarrow$: Let U be a neighborhood of x. From the discussion above we may assume U is of the form

$$U = \bigcap_{i \in J} \varphi_i^{-1}(V_i),$$

$J \subset I$ finite, $\varphi_i(x) \in V_i \in \tau_i$. Since $\varphi_i(x_n) \to \varphi_i(x)$ $\forall i \in I \Rightarrow$ for $i \in J$ $\exists N_i \in \mathbb{N} : \varphi_i(x_n) \in V_i$ $\forall n \geq N_i$. Choose $N := \max_{i \in J}(N_i) < \infty$

$$\Rightarrow \varphi_i(x_n) \in V_i \quad \forall i \in J, \forall n \geq N$$

$$\Rightarrow x_n \in U \quad \forall n \geq N.$$

□

Proposition 7.3. *Let Z be a topological space, $\psi : Z \to X$. Then*

$$\psi \text{ is continuous} \iff \varphi_i \circ \psi : Z \to Y_i \text{ is continuous } \forall i \in I.$$

Proof. "$\Rightarrow$": Simple: use that compositions of continuous functions are continuous.
"$\Leftarrow$": Need to show: $\psi^{-1}(U)$ is open (in Z) $\forall$ open set U in X. U has the form

$$U = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \varphi_i^{-1}(V_i), \quad V_i \in \tau_i$$

$$\begin{aligned}\psi^{-1}(U) &= \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \psi^{-1}(\varphi_i^{-1}(V_i)) \\ &= \underbrace{\bigcup_{\text{arbitrary}} \underbrace{\bigcap_{\text{finite}} \underbrace{(\varphi_i \circ \psi)^{-1}(V_i)}_{\text{open in } Z}}_{\text{is open}}}_{\text{is open!}}\end{aligned}$$

so $\psi^{-1}(U)$ is open in Z, so ψ is continuous. □

7.2 The weakest topology $\sigma(E, E^*)$

Let E be a Banach space, E^* the dual, so E has a norm $\|\cdot\| = \|\cdot\|_E$, $f \in E^*$ are continuous linear functionals on E. For $f \in E^*$ let

$$\varphi_f : \begin{cases} E \to \mathbb{F} \\ x \mapsto \varphi_f(x) := f(x) \end{cases}$$

Take $I = E^*, Y_i = \mathbb{F}, X = E$ with the usual topology on $\mathbb{R}$, resp. $\mathbb{C}$.

Definition 7.4. *The weak topology $\sigma(E, E^*)$ on E is the coarsest (smallest) topology associated with the collection $(\varphi_f)_{f \in E^*}$ in the sense of Section 7.1.*

Note: Since every map $\varphi_f = f$ is continuous linear functional the weak topology is weaker (it contains fewer open sets) than the usual topology on E induced by the norm on E!

Proposition 7.5. *The weak topology $\sigma(E, E^*)$ is Hausdorff (i.e., it separates points).*

Proof. Let $x_1, x_2 \in E, x_1 \neq x_2$. We need to construct open sets $\mathcal{O}_1, \mathcal{O}_2 \in \sigma(E, E^*)$ with $x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2, \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$.
By Hahn-Banach (2nd geometric form) we can strictly separate $\{x_1\}, \{x_2\}$ by some $f \in E^*$, i.e., $\exists \alpha \in \mathbb{R}$

$$Ref(x_1) < \alpha < Ref(x_2).$$

Set

$$\mathcal{O}_1 := \{x \in E | Ref(x) < \alpha\} = \varphi_f^{-1}((-\infty, \alpha) + i\mathbb{R}) \in \sigma(E, E^*)$$
$$\mathcal{O}_2 := \{x \in E | Ref(x) > \alpha\} = \varphi_f^{-1}((\alpha, +\infty) + i\mathbb{R}) \in \sigma(E, E^*).$$

Clearly $x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2, \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. □

Proposition 7.6. *Let $x_0 \in E$. Given $\varepsilon > 0$ and finitely many $f_1, \ldots, f_k \in E^*$, and let*

$$V := V(f_1, \ldots f_k, \varepsilon) := \{x \in E | \ |f_i(x - x_0)| < \varepsilon, \forall i = 1, \ldots, k\}.$$

Then V is a neighborhood of x_0 in $\sigma(E, E^*)$ and we get a basis of neighborhoods of x_0 in $\sigma(E, E^*)$ by varying $\varepsilon > 0, k \in \mathbb{N}$, and $f_1, \ldots f_k \in E^*$.

Proof.

$$x_0 \in V = \bigcap_{i=1}^{k} \varphi_{f_i}^{-1}(\{z \in \mathbb{C} | \ |z - \alpha_i| < \varepsilon\}) \in \sigma(E, E^*), \quad \alpha_i := f_i(x_0), \text{ is open!}$$

Conversely, let $x_0 \in U \in \sigma(E, E^*)$. By definition of $\sigma(E, E^*)$, U contains an open set $W \ni x_0$ of the form

$$W = \bigcap_{\text{finite}} \varphi_{f_i}^{-1}(V_i),$$

V_i neighborhood of $f_i(x_0) = \alpha_i$ in $\mathbb{F}$.

$$\Rightarrow \exists \varepsilon > 0 : \quad \{z \in \mathbb{C} | \ |z - \alpha_i| < \varepsilon\} \subset V_i \quad \forall i = 1, \ldots k$$

so $x_0 \in V \subset W \subset U$. □

Notation: If $(x_n)_n \subset X$ converges to x in the weak topology $\sigma(E, E^*)$, we write $x_n \rightharpoonup x$ (or $x_n \rightharpoonup x$ in $\sigma(E, E^*)$, or $x_n \rightharpoonup x$ weakly in $\sigma(E, E^*)$, or $x_n \rightharpoonup x$ weakly). We say that $x_n \to x$ strongly if $\|x_n - x\| \to 0$ (usual convergence in E).

Proposition 7.7. *Let $(x_n)_n \subset E$ be a sequence. Then*

(a) $x_n \rightharpoonup x$ weakly $\iff f(x_n) \to f(x) \ \forall f \in E^*$.

(b) If $x_n \to x$ strongly then $x_n \rightharpoonup x$ weakly.

(c) If $x_n \rightharpoonup x$ weakly, then $(\|x_n\|)_n$ is bounded in $\mathbb{R}$ and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

(d) If $x_n \rightharpoonup x$ weakly and $f_n \to f$ strongly in E^* (i.e., $\|f_n - f\|_{E^*} \to 0$) then $f_n(x_n) \to f(x)$.

Proof. (a) Follows from the definition of $\sigma(E, E^*)$ and Proposition 7.2.

(b) By (a)

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq \|f\|_{E^*}\|x_n - x\|_E \to 0.$$

(c) Note that $\forall f \in E^*, (f(x_n))_n \subset \mathbb{F}$ is bounded. Therefore, by the uniform boundedness principle

$$\infty > \sup_{n\in N} \underbrace{\sup_{f\in E^*, \|f\|_{E^*}\leq 1} |f(x_n)|}_{=\|x_n\|_E} = \sup_{n\in N} \|x_n\|_E.$$

$$|f(x)| \leftarrow |f(x_n)| \leq \|f\|_{E^*}\|x_n\|_E \leq \|x_n\|_E \quad \text{if } \|f\|_{E^*} \leq 1.$$

$$\Rightarrow |f(x)| \leq \liminf_{n\to\infty} \|x_n\|_E \quad \forall f \in E^*, \|f\|_{E^*} \leq 1$$

$$\Rightarrow \|x\|_E = \sup_{f\in E^*, \|f\|_{E^*}\leq 1} |f(x)| \leq \liminf_{n\to\infty} \|x_n\|_E.$$

(d) Note that by (a) and (c)

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &\leq \|f_n - f\|_{E^*}\|x_n\| + |f(x_n - x)| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

Proposition 7.8. *If E is finite-dimensional then $\sigma(E, E^*)$ and the usual topology are the same, so a sequence $(x_n)_n$ converges weakly $\Leftrightarrow$ $(x_n)_n$ converges strongly.*

Proof. Since $\sigma(E, E^*)$ contains fewer open sets than the strong topology it is enough to show that every (strongly) open set is weakly open.
Let $x_0 \in E$ and U strongly open with $x_0 \in U$. Need to find $f_1, \ldots, f_k \in E^*, \varepsilon > 0$ with

$$V := V(f_1, \ldots, f_k, \varepsilon) = \{x \in E |\ |f_i(x - x_0)| < \varepsilon \text{ for all } i = 1, \ldots, k\} \subset U.$$

Let $r > 0$ such that $B_r(x_0) \subset U$. Pick a basis $e_1, e_2, \ldots, e_k$ in E such that $\|e_i\| = 1$ for all $i = 1, \ldots k$. Note that

$$x = \sum_{j=1}^{k} x_j e_j \quad \text{and} \quad x \mapsto x_j =: f_j(x)$$

are continuous linear functionals on E. Also

$$\begin{aligned}\|x - x_0\| &= \|\sum_{j=1}^{k} f_j(x - x_0)e_j\| \\ &\leq \sum_{j=1}^{k} |f_j(x - x_0)|\|e_j\| \\ &= \sum_{j=1}^{k} |f_j(x - x_0)| \leq k \cdot \varepsilon \quad \forall x \in V.\end{aligned}$$

Choose $r = \frac{\varepsilon}{k}$ to get $V \subset U$. □

Remark 7.9. *Weakly open (resp. closed) sets are always open (resp. closed) in the strong topology! If E is infinite-dimensional, the weak topology $\sigma(E, E^*)$ is strictly coarser (smaller) than the strong topology.*

Example. *Let E be infinite-dimensional. The unit sphere*

$$S := \{x \in E|\ \|x\| = 1\} \quad \Rightarrow \quad \overline{S}^{\sigma(E,E^*)} = B_E = \{x \in E|\ \|x\| \leq 1\}!$$

Proof. Step 1: $\{x \in E|\ \|x\| \leq 1\} \subset \overline{S}^{\sigma(E,E^*)}$.
Indeed, let $x_0 \in V \subset \sigma(E, E^*)$. Need to show that $V \cap S \neq \emptyset$!.
By Proposition 7.6, we may assume

$$V = \{x \in E|\ |f_i(x - x_0)| < \varepsilon \quad \forall i = 1, \ldots, k\}$$

for some $\varepsilon > 0, f_1, \ldots, f_k \in E^*$.
Claim: $\exists y_0 \in E \setminus \{0\}$ with $f_i(y_0) = 0\ \forall i = 1, \ldots, k$.
If not, the map

$$\varphi : \begin{cases} E \to \mathbb{F}^k \\ x \mapsto \varphi(x) := (f_1(x), f_2(x), \ldots, f_k(x)) \end{cases}$$

is injective (why?) and hence φ would be injective and surjective from E onto $\varphi(E) \subset \mathbb{F}^k$. Since $\varphi(E) \subset \mathbb{F}^k$ is a Banach space, the inverse mapping theorem would give that φ and φ^{-1} are continuous so E is homeomorphic to a finite-dimensional space, hence E would be finite-dimensional. So the claim is true.
Note that $x_0 + ty_0 \in V$ for all $t \in \mathbb{R}$. Since $g(t) := \|x_0 + ty_0\|$ is continuous on $[0, \infty)$, $g(0) = \|x_0\| < 1, \lim_{t\to\infty} g(t) = \infty$, there exists $t_0 > 0 : \|x_0 + ty_0\| = 1 \Rightarrow x_0 + t_0y_0 \subset S \cap V$. By Step 1 we see

$$S \subset B_E \subset \overline{S}^{\sigma(E,E^*)}. \tag{$*$}$$

Step 2: B_E is closed in the weak topology.
Indeed,

$$B_E = \bigcap_{f\in E^*, \|f\|_{E^*}\leq 1} \underbrace{\{x \in E|\ |f(x)| \leq 1\}}_{\text{weakly closed}} \quad \text{is weakly closed.}$$

By $(*)$ and Step 2: $B_E = \overline{S}^{\sigma(E,E^*)}$ since $\overline{S}^{\sigma(E,E^*)}$ is the smallest weakly closed set containing B_E and B_E is weakly closed. □

Example. *The unit ball $U = \{x \in E | \ \|x\| < 1\}$, E infinite-dimensional, is* <u>*not*</u> *weakly open.*
Indeed, if U were weakly open then $U^c = \{x \in E | \ \|x\| \geq 1\}$ is weakly closed and hence

$$S = B_E \cap U^c$$

is weakly closed which by the previous Example it is not!

7.3 Weak topology and convex sets

Recall that every weakly closed set is strongly closed, but the converse is false if E is infinite-dimensional.
But: convex + strongly closed $\Rightarrow$ weakly closed.

Theorem 7.10. *Let $C \subset E$ be convex. Then C is closed if and only if C is weakly closed.*

Proof. "$\Leftarrow$": Clear since C^c is weakly open, hence open.
"$\Rightarrow$": Need to check that C^c is weakly open. Let $x_0 \notin C$. By Hahn-Banach, there exists a closed hyperplane which strictly separates $\{x_0\}$ and C, i.e., there exists $f \in E^*, \alpha \in \mathbb{R}$ such that

$$Ref(x_0) < \alpha < Ref(y) \quad \forall y \in C.$$

Set

$$V := \{x \in E | Ref(x) < \alpha\} \in \sigma(E, E^*).$$

Then $x_0 \in V, V \cap C = \emptyset$ so $V \subset C^c$. □

Remark 7.11. *The above proof shows that $C = \bigcap H_c$ where the intersection is over all closed half-spaces H_C which contain C.*

Corollary 7.12 (Mazur)**.** *Assume that $x_n \rightharpoonup x$ weakly. Then there exists a sequence $(y_n)_n$ of convex combinations of x_n which converges strongly to x.*

Proof. Let $C := conv(\bigcup_{l=1}^{\infty}\{x_l\})$ be the convex hull of x_n. Since x belongs to the weak closure of $\bigcup_{l=1}^{\infty}\{x_l\}$, it also belongs to the weak closure of C! By Theorem 7.10 we get $x \in \bar{C}$, the strong closure of C! □

Corollary 7.13. *Assume $\varphi : E \to (-\infty, +\infty]$ is convex and lower semi-continuous (l.s.c) in the strong topology. Then φ is l.s.c. in the weak topology.*

Proof. φ is (strongly) l.s.c. if for every sequence $(x_n)_n \subset E, x_n \to x$ one has

$$\liminf_{n\to\infty} \varphi(x_n) \geq \varphi(x)$$

and similarly for weakly l.s.c. (replace $x_n \to x$ by $x_n \rightharpoonup x$).
In terms of the level sets:

Lemma. *$\varphi : E \to (-\infty, +\infty]$ is strongly (resp. weakly) l.s.c. if for all $\lambda \in \mathbb{R}$ the sets*

$$A_\lambda := \{x \in E | \varphi(x) \leq \lambda\}$$

are strongly (resp. weakly) closed.

Proof. If φ is strongly l.s.c and $x_n \in A_\lambda$ with $x_n \to x$, then

$$\varphi(x) \leq \liminf_{n\to\infty} \underbrace{\varphi(x_n)}_{\leq\lambda} \leq \lambda$$

so $x \in A_\lambda$, i.e., A_λ is closed.

For the converse, assume that φ is not l.s.c. at some point x but A_λ is closed $\forall\lambda \in \mathbb{R}$. So there exists a sequence $(x_n)_n \subset E, x_n \to x$ and

$$\liminf_{n\to\infty} \varphi(x_n) < \varphi(x).$$

Thus there exists a subsequence, also called $(x_n)_n$, and $\lambda \in \mathbb{R}$ such that

$$\varphi(x_n) < \lambda < \varphi(x) \quad \forall n \in \mathbb{N}.$$

But then $x_n \in A_\lambda\ \forall n \in \mathbb{N}$ and since $x_n \to x$ and A_λ is closed, also $x \in A_\lambda$, i.e., $\varphi(x) \leq \lambda$, a contradiction.

For the statement with strongly replaced by weakly, just replace $x_n \to x$ by $x_n \rightharpoonup x$ in the proof. □

Continuing the proof of the Corollary, we have

$$A_\lambda = \{x \in E | \varphi(x) \leq \lambda\}$$

is closed, since φ is strongly l.s.c. Since φ is convex, we also have that A_λ is convex (why?)! So A_λ is convex and strongly closed and by Theorem 7.10 it is weakly closed! □

Example. *$\varphi(x) = \|x\|$ is convex and strongly continuous so it is weakly l.s.c. Hence if $x_n \rightharpoonup x$ weakly, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$ (compare with Proposition 7.7).*

Theorem 7.14. *Let E, B be Banach spaces and $T : E \to B$ linear. Then T is continuous in the strong topologies on E and B if and only if T is continuous in the weak topologies on E and B.*

Proof. "$\Rightarrow$": By Proposition 7.3, we need to show that for any $f \in B^*$ the composition $f \circ T$, i.e., the map $x \mapsto f(Tx)$ is continuous from $(E, \sigma(E, E^*))$ to $\mathbb{F}$.
Since $x \mapsto f(Tx) \in E^*$ it is automatically also continuous w.r.t. $\sigma(E, E^*)$!
"$\Leftarrow$": Assume $T : (E, \sigma(E, E^*)) \to (B, \sigma(B, B^*))$ is continuous. Then

$$G(T) = \{(x, Tx) | x \in E\} \subset E \times B$$

is closed in $E \times B$ equipped with the product topology $\sigma(E, E^*) \times \sigma(B, B^*) = \sigma(E \times B, (E \times B)^*)$. So $G(T)$ is weakly closed, but then also strongly closed in $E \times B$. By the Closed graph theorem it follows that $T : E \to B$ is continuous in the strong topology. □

7.4 The weak$*$ topology $\sigma(E^*, E)$

Consider the dual space E^* of a normed vector space E. So far, we have two topologies on E^*:

(a) The usual (strong) topology associated to the norm on E^*, $\|f\|_{E^*} := \sup_{\|x\|_E \leq 1} |f(x)|$.

(b) The weak topology $\sigma(E^*, E^{**})$, where $E^{**} = (E^*)^*$ is the dual of E^*, from the last sections.

Note that we can always consider E as a subset of $E^{**} = \{$ continuous linear functionals on $E^*\}$ by the following device: Given $x \in E$ let $\varphi_X : E^* \to \mathbb{F}$ be defined by

$$\varphi_x(f) := f(x).$$

Then $\varphi_x \in E^{**}$ corresponds to $x \in E$ and $x \mapsto \varphi_x$ is injective since if $\varphi_{x_1} = \varphi_{x_2}$ then for all $f \in E^*$ one has

$$f(x_1) = \varphi_{x_1} = \varphi_{x_2} = f(x_2)$$

and since E^* separates the points in E this means $x_1 = x_2$! So the map $x \mapsto \varphi_x$ yields an injection of E into E^{**}.

Definition 7.15. *The weak$*$ topology $\sigma(E^*, E)$ is the smallest topology on E^* associated with the family $(\varphi_x)_{x \in E}$, i.e., it is the smallest topology on E^* which makes all the maps $\varphi_x : E^* \to \mathbb{F}, x \in E$, continuous.*

Remark 7.16.
- *Since $E \subset E^{**}$ it is clear that $\sigma(E^*, E)$ contains fewer open sets than the weak topology $\sigma(E^*, E^{**})$ which in turn has fewer open sets that the strong topology on E^*.*
- *The reason why one wants to study these different notions of weak topologies is that the fewer open sets a topology has, the more sets are compact! Since compact sets are fundamentally important – e.g., in the proof of existence of minimizers – it is easy to understand the importance of the weak$*$ topology.*

Proposition 7.17. *The weak$*$ topology is Hausdorff.*

Proof. Given $f_1, f_2 \in E^*$ with $f_1 \neq f_2$, there exists $x \in E$ such that $f_1(x) \neq f_2(x)$ (this DOES NOT use Hahn-Banach, but just the fact that $f_1 \neq f_2$!).
W.l.o.g., we can assume that $Re\ f_1(x) \neq Re\ f_2(x)$. If not, then $Im\ f_1(x) \neq Im\ f_2(x)$ and hence

$$Re(-if_1(x)) = Im\ f_1(x) \neq Im\ f_2(x) = Re(-if_2(x))$$

so consider $-if_1, -if_2$ instead of f_1 and f_2.
W.l.o.g., $Re\ f_1(x) < Re\ f_2(x)$ and choose $\alpha \in \mathbb{R} : Re\ f_1(x) < \alpha < Re\ f_2(x)$. Set

$$\mathcal{O}_1 := \{f \in E^* | Ref(x) < \alpha\} = \varphi_x^{-1}([-\infty, \alpha) + i\mathbb{R})$$
$$\mathcal{O}_2 := \{f \in E^* | Ref(x) > \alpha\} = \varphi_x^{-1}((\alpha, \infty) + i\mathbb{R})$$

Then $\mathcal{O}_1, \mathcal{O}_2 \in \sigma(E^*, E), f_1 \in \mathcal{O}_1, f_2 \in \mathcal{O}_2$ and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. $\square$

Proposition 7.18. *Let $f_0 \in E^*, n \in \mathbb{N}, \{x_1, x_2, \ldots, x_n\} \subset E$ and $\varepsilon > 0$. Consider*

$$V = V(x_1, \ldots, x_n, \varepsilon) := \{f \in E^* | \, |(f - f_0)(x_j)| < \varepsilon \text{ for all } j = 1, \ldots, n\}.$$

Then V is a neighborhood of f_0 in $\sigma(E^*, E)$. Moreover, we obtain a basis of neighborhoods of f_0 in $\sigma(E, E^*)$ by varying $\varepsilon > 0, n \in \mathbb{N}$, and $x_1, \ldots, x_n \in E$.

Proof. A literal transcription of the proof of Proposition 7.6. □

Notation: If a sequence $(f_n)_n \subset E^*$ converges to f in the weak$*$ topology, we write $f_n \overset{*}{\rightharpoonup} f$.

To avoid confusion, we sometimes emphasize "$f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E)$", "$f_n \rightharpoonup f$ in $\sigma(E^*, E^{**})$" and "$f_n \to f$ strongly".

Proposition 7.19. *Let $(f_n)_n \subset E$. Then*

(a) $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E) \iff f_n(x) \to f(x), \forall x \in E$ (i.e., convergence of functionals in $\sigma(E^*, E)$ is the same as pointwise convergence of f_n to f!).

(b) If $f_n \to f$ strongly, then $f_n \rightharpoonup f$ in $\sigma(E^*, E^{**})$.
If $f_n \rightharpoonup f$ in $\sigma(E^*, E^{**})$, then $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E)$.

(c) If $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E)$, then $(\|f_n\|)_n$ is bounded and $\|f\| \leq \liminf \|f_n\|$.

(d) If $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E)$ and if $x_n \to x$ strongly in E, then $f_n(x) \to f(x)$.

Proof. Copy the proof of Proposition 7.7. □

Remark 7.20. *When E is finite-dimensional, then the three topologies (strong, weak, weak$*$) on E^* coincide! Indeed, then the canonical injection $J : E \to E^{**}$ given by $x \mapsto \varphi_x, \varphi_x(f) := f(x), f \in E^*$ is surjective (since $dimE = dimE^{**}$) and therefore $\sigma(E^*, E) = \sigma(E^*, E^{**})$.*

The main result about compactness in the weak$*$ topology is the famous

Theorem 7.21 (Banach-Alaoglu-Bourbaki)**.** *The closed unit ball*

$$B_{E^*} := \{f \in E^* | \, \|f\|_{E^*} \leq 1\}$$

is compact in the weak$*$ topology $\sigma(E^*, E)$.

Note: This compactness property is the most essential property of the weak$*$ topology!

Proof. We will reformulate the problem slightly: Consider the cartesian product

$$Y := \mathbb{F}^E = \{\text{maps } \omega : E \to \mathbb{F}\} = (\omega(x))_{x \in E} \text{ with } \omega(x) \in \mathbb{F}.$$

We equip Y with the standard product topology, i.e., the smallest topology on Y such that the collection of maps

$$\mathbb{F}^E = Y \ni \omega \mapsto \omega(x) \in \mathbb{F}, x \in E$$

is continuous for all $x \in E$. This is the same as the topology of pointwise convergence, i.e., $(\omega_n)_n \subset Y$ converges to ω if $\forall x \in E, \omega_n(x) \to \omega(x)$ (see Munkres: Topology, A First Course, Prentice Hall, 1975 or Dixmier: General Topology, Springer 1984, or Knapp: Basic Real Analysis, Birkhäuser, 2005).