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Lectures Notes in Functinoal Analysis

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A very useful fact on product topology:

Theorem (Tychonov's theorem). *An arbitrary product of compact spaces is compact in the product topology.*

Proof. See the above books. □

Note: E^* consists of very special maps from E to $\mathbb{F}$, namely the continuous linear maps. So we may consider E^* as a subset of Y!

More precisely, let

$$\Phi : E^* \to Y$$

be the canonical injection from E^* to Y given by

$$\Phi(f) := (\Phi(f)_x)_{x\in E} = (f(x))_{x\in E}.$$

Clearly, Φ is continuous from E^* into Y. To see this, simply use Proposition 7.3 and note that for each fixed $x \in E$, the map

$$E^* \ni f \mapsto (\Phi(f))_x = f(x)$$

is continuous!

Since the inverse $\Phi^{-1} : \Phi(E^*) \to E^*$ is given by

$$\omega \mapsto (E \ni x \mapsto \Phi^{-1}(\omega)(x) := \omega(x)),$$

one sees that $\Phi^{-1} : Y \supset \Phi(E^*) \to E^*$ is also continuous when Y is given the product topology. Indeed, using Proposition 7.3 again, it is enough to check, for each fixed $x \in E$, that the map $\omega \mapsto \Phi^{-1}(\omega)(x) := \omega(x))$ is continuous on $\Phi(E^*) \subset Y$. But this is obvious, since Y is given the product topology, so if $\omega_n \to \omega$ in Y then $\omega_n(x) \to \omega(x)$ for all $x \in E$, so

$$\Phi^{-1}(\omega_n)(x) = \omega_n(x) \to \omega(x) = \Phi^{-1}(\omega)(x) \quad \text{as } n \to \infty.$$

Upshot: Φ is a homeomorhism from E^* onto $\Phi(E^*) \subset Y$ where E^* is given the weak$*$ topology $\sigma(E^*, E)$ and Y is given the product topology.

Note: $\Phi(B_{E^*}) = K$, where the set $K \subset Y$ is given by

$$\begin{aligned} K = \{\omega \in Y |\ |\omega(x)| \le \|x\|_E, \omega \text{ is linear, i.e.,} \\ \omega(x+y) = \omega(x) + \omega(y) \text{ and} \\ \omega(\lambda x) = \lambda\omega(x)\ \forall \lambda \in \mathbb{F}, x, y \in E\}. \end{aligned}$$

Now we only have to check that K is a compact subset of Y!

We can write $K = K_1 \cap K_2$ where

$$K_1 = \{\omega \in Y |\ |\omega(x)| \le \|x\|_E\ \forall x \in E\}$$

and

$$K_2 := \Phi(E^*) = \{\omega \in Y | \omega \text{ is linear}\}.$$

Note that K_1 can be written as

$$K_1 = \prod_{x\in E} [-\|x\|, \|x\|] \subset \mathbb{R}^E \quad \text{if } \mathbb{F} = \mathbb{R}$$

or

$$K_1 = \prod_{x\in E} \{z \in \mathbb{C} |\ |z| \leq \|x\|\} \subset \mathbb{C}^E \quad \text{if } \mathbb{F} = \mathbb{C}$$

and by Tychonov's theorem K_1 is a compact subset of Y!

So we only have to show that K_2 is closed (since the intersection of a closed set and a compact set is compact!).
Let

$$B_{x,y,\lambda_1,\lambda_2} := \{\omega \in Y | \omega(\lambda_1 x + \lambda_2 y) - \lambda_1\omega(x) - \lambda_2\omega(y) = 0\}$$

which are closed subsets of Y, since if $\omega_n \in B_{x,y,\lambda_1,\lambda_2}$ then, if $\omega_n \to \omega$ in Y, then

$$\begin{aligned} 0 =& \omega_n(\lambda_1 x + \lambda_2 y) - \lambda_1\omega_n(x) - \lambda_2\omega_n(y) \\ &\to \omega(\lambda_1 x + \lambda_2 y) - \lambda_1\omega(x) - \lambda_2\omega(y) \quad \text{as } n \to \infty \end{aligned}$$

so $\omega \in B_{x,y,\lambda_1,\lambda_2}$.

So

$$K_2 := \bigcap_{x,y\in E,\lambda_1,\lambda_2\in\mathbb{F}} B_{x,y,\lambda_1,\lambda_2}$$

is closed in Y!

Hence $K = K_1 \cap K_2$ is compact and so $B_{E^*} = \Phi^{-1}(K)$ is compact in E^* w.r.t $\sigma(E^*, E)$. □

7.5 Reflexive spaces

Definition 7.22. *Let E be a Banach space and $J : E \to E^{**}$ the canonical injection from E into E^{**} given by*

$$(J(x))(f) := \varphi_x(f) := f(x) \quad \forall x \in E, f \in E^*.$$

The space E is ***reflexive*** *if J is surjective, i.e., $J(E) = E^{**}$.*

Note: When E is reflexive, E^{**} is usually identified with E!

Remark 7.23. *(a) Finite-dimensional spaces are reflexive (since $dimE = dimE^* = dimE^{**}$).*
Later we will see that L^p and l^p are reflexive if $1 < p < \infty$.

(b) Every Hilbert space is reflexive.

(c) L^1, L^∞, l^1 and l^∞ are not reflexive.

$C(K)$ = space of continuous functions on an infinite compact metric space K

is not reflexive.

(d) *It is essential to use the canonical injection J in the definition of reflexive spaces. See R.C. James: A non-reflexive Banach space isometric with its second conjugate, Proc. Nat. Acad. Sci USA* **37** *(1951), pp 174-177, for a non-reflexive Banach space for which E is isometric to E^{**}.*

Theorem 7.24 (Kakutani)**.** *Let E be a Banach space. Then E is reflexive if and only if $B_E = \{x \in E| \, \|x\| \le 1\}$ is compact in the weak topology $\sigma(E, E^*)$.*

Proof. "⇒": Here $J(B) = B_{E^{**}}$ by assumption. By Theorem 7.21 we know that $B_{E^{**}}$ is compact in the weak∗ topology $\sigma(E^{**}, E^*)$. So it is enough to check that $J^{-1} : E^{**} \to E$ is continuous when E^{**} is equipped with the weak∗ topology $\sigma(E^{**}, E^*)$ and E is equipped with the weak topology $\sigma(E, E^*)$.
But a map $J^{-1} : E^{**} \to E$ is continuous when E is given the weak topology if and only if $\forall f \in E^*$ the map $\xi \mapsto f(J^{-1}(\xi))$ is continuous.
Note that $f(J^{-1}(\xi)) = \xi(f), \xi \in E^{**}$ but for fixed f the map $E^{**} \ni \xi \mapsto \xi(f)$ is continuous on E^{**} with the weak∗ topology $\sigma(E^{**}, E^*)$! So J^{-1} is continuous and $B_E = J^{-1}(B_{E^{**}})$ is compact.
"⇐:" We need the following two lemmata

Lemma 7.25. *Let E be a Banach space, $f_1, \dots f_k \in E^*$ and $\gamma_1, \dots, \gamma_k \in \mathbb{F}$. Then*

(a) $\forall \varepsilon \; \exists x_\varepsilon \in E$ *with* $\|x_\varepsilon\| \le 1$ *and* $|f_l(x_\varepsilon) - \gamma| < \varepsilon \; \forall l = 1, \dots, k$

is equivalent to

(b) $|\sum\limits_{l=1}^{k} \beta_l \gamma_l| \le \| \sum\limits_{l=1}^{k} \beta_l f_l\| \; \forall \beta_1, \dots, \beta_k \in \mathbb{F}$.

Proof. Only for $\mathbb{F} = \mathbb{C}$.
"$(a) \Rightarrow (b)$": Fix $\beta_1, \dots, \beta_k \in \mathbb{C}$, $S := \sum\limits_{l=1}^{k} |\beta_l|$. By (a) we have

$$|\sum_{l=1}^{k} \beta_l f_l(x_\varepsilon) - \sum_{l=1}^{k} \beta_l \gamma_l| \le \varepsilon S$$

and hence

$$\begin{aligned} |\sum_{l=1}^{k} \beta_l \gamma_l| &\le |\sum_{l=1}^{k} \beta_l f_l(x_\varepsilon)| + \varepsilon S \\ &\le \|\sum_{l=1}^{k} \beta_l f_l\|_{E^*} \|x_\varepsilon\|_E + \varepsilon S \quad \forall \varepsilon > 0. \end{aligned}$$

"$(b) \Rightarrow (a)$": We will show that not $(b) \Rightarrow$ not (a):
Let $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{C}^k$ and let $\varphi : E \to \mathbb{C}^k$ be given by

$$\varphi(x) := (f_1(x), f_2(x), \dots, f_k(x)).$$

Then (a) can be rephrased as follows

$$\gamma \in \overline{\varphi(B_E)} \quad \text{(closure in } \mathbb{C}^k)$$

and not (a) means $\gamma \notin \overline{\varphi(B_E)}$, i.e., $\{\gamma\}$ and $\overline{\varphi(B_E)}$ can be striclty separated in $\mathbb{C}^k$ by a hyperplane, i.e., there exist $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k = (\mathbb{C}^k)^*$ and $\alpha \in \mathbb{R}$ such that for all $x \in B_E$

$$\begin{aligned} Re(\beta(\varphi(x))) = Re(\beta \cdot \varphi(x)) = Re \sum_{l=1}^{k} \beta_l f_l(x) \\ < \alpha < Re(\beta \cdot \gamma) = Re \sum_{l=1}^{k} \beta_l \gamma_l. \end{aligned}$$

Therefore (take sup over $\|x\| \leq 1$)

$$\| \sum_{l=1}^{k} \beta_l f_l \| \leq \alpha < Re \sum_{l=1}^{k} \beta_l \gamma_l \leq | \sum_{l=1}^{k} \beta_l \gamma_l |,$$

i.e., not (b) is true! □

Lemma 7.26. *Let E be a Banach space. Then $J(B_E)$ is dense in $B_{E^{**}}$ w.r.t. the weak* topology $\sigma(E^{**}, E^*)$ on E^{**}. Consequently, $J(E)$ is dense in E^{**} w.r.t. the weak* topology $\sigma(E^{**}, E^*)$ on E^{**}.*

Proof. Let $\xi \in B_{E^{**}}$ and V be a neighborhood of ξ in $\sigma(E^{**}, E^*)$. Need to show $V \cap J(B_E) \neq \emptyset$. As usual, we may assume that V is of the form

$$V = \{\eta \in E^{**} | \, |(\eta - \xi)(f_j)| < \varepsilon, \ \forall j = 1, \ldots, k\}$$

for some $f_1, \ldots f_k \in E^*, \varepsilon > 0$.
We have to find $x \in B_E$ with $J(x) \in V$, i.e.,

$$|f_l(x) - \xi(f_l)| < \varepsilon \quad \forall l = 1, \ldots k.$$

Set $\gamma_l := \xi(f_l)$. By Lemma 7.25 we need to check

$$| \sum_{l=1}^{k} \beta_l \gamma_l | \leq \| \sum_{l=1}^{k} \beta_l f_l \|$$

but this is clear since

$$\sum_{l=1}^{k} \beta_l \gamma_l = \sum_{l=1}^{k} \beta_l \xi(f_l) = \xi(\sum_{l=1}^{k} \beta_l f_l) \quad (\xi \in E^{**})$$

so

$$| \sum_{l=1}^{k} \beta_l \gamma_l | = |\xi(\sum_{l=1}^{k} \beta_l f_l) \leq \| \sum_{l=1}^{k} \beta_l f_l \|_{E^*} \underbrace{\|\xi\|_{E^{**}}}_{\leq 1}.$$

□

Remark 7.27. *$J(B_E)$ is always closed in $B_{E^{**}}$ in the strong topology on E^{**}! Indeed, if $\xi_n = J(x_n) \to \xi$ then, since J is an isometry, x_n must be Cauchy in B_E, so $x_n \to x$ and $\xi = J(x)$. Thus $J(B_E)$ is not dense in $B_{E^{**}}$ in the strong topology unless $J(B_E) = B_{E^{**}}$, i.e., E is reflexive!*

Continuing the proof of Theorem 7.24 "$\Leftarrow$":
The canonical injection $J : E \to E^{**}$ is always continuous from $\sigma(E, E^*)$ into $\sigma(E^{**}, E^*)$ since for fixed $f \in E^*, x \mapsto (Jx)(f) = f(Jx)$ is continuous w.r.t. $\sigma(E, E^*)$. Assuming that B_E is weakly compact (i.e., in $\sigma(E, E^*)$ topology) we see that $J(B_E)$ is compact and thus closed in E^{**} w.r.t. $\sigma(E^{**}, E^*)$.
But by Lemma 7.26, $J(B_E)$ is dense in $B_{E^{**}}$ for the same topology! Therefore $J(B_E) = B_{E^{**}}$, hence $J(E) = E^{**}$, i.e., E is reflexive. □

Theorem 7.28. *Assume that E is a reflexive Banach space and $(x_n)_n \subset E$ a bounded sequence. Then there exists a subsequence (x_{n_l}) that converges weakly.*

Remark 7.29. *A result of Eberlein-Šmulian says that if E is a Banach space such that every bounded sequence has a weakly convergent subsequence then E is reflexive! (See Holmes: Geometric Functional Analysis and its Applications, Springer, 1975).*

Proposition 7.30. *Let E be a reflexive Banach space and $M \subset E$ a closed linear subspace of E. Then M is reflexive.*

Proof. M, equipped with the norm from E has a-priori two distinct weak topologies:

(a) the topology induced by $\sigma(E, E^*)$

(b) its own weak topology $\sigma(M, M^*)$.

Fact: these two topologies are the same since by Hahn-Banach, every continuous linear functional on M is the restriction of a continuous linear functional on E!

By Theorem 7.24 we need to check that B_M is compact in the weak topology $\sigma(M, M^*)$, or equaivalently, in the topology $\sigma(E, E^*)$! We know that B_E is compact in the weak topology and since M is (strongly) closed and convex it is also weakly closed by Theorem 7.10. So $B_M = M \cap B_E$ is weakly compact! □

Corollary 7.31. *A Banach space E is reflexive if and only if E^* is reflexive.*

Proof. "$\Rightarrow$": Roughly: $E^{**} = E \Rightarrow E^{***} = E^*$.
More precisely, let $J : E \to E^{**}$ be the canonical isometry. Let $\varphi \in E^{***}$. The map

$$x \mapsto f_\varphi(x) := \varphi(J(x))$$

is a continuous linear functional on E, so $f \in E^*$.
Note:

$$\varphi(J(x)) = f(x) = (J(x))(f) \quad \forall x \in E, J(x) \in E^{**}. \qquad (*)$$

By assumption $J : E \to E^{**}$ is surjective so for every $\xi \in E^{**}$ $\exists x \in E, \xi = J(x)$. So $(*)$ yields

$$\varphi(\xi) = \xi(f) \quad \forall \xi \in E^{**},$$

i.e., the canonical injection $E^* \to E^{***}$ is surjective.
"$\Leftarrow$": Let E^* be reflexive. By "$\Rightarrow$" above we know that E^{**} is reflexive. Since $J(E) \subset E^{**}$ is a closed subspace in the strong topology, Theorem 7.30 yields that $J(E)$ is reflexive. Thus E is reflexive! □

Corollary 7.32. *Let E be a reflexive Banach space, $K \subset E$ a bounded, closed and convex subset. Then K is compact in the weak topology $\sigma(E, E^*)$.*

Proof. By Theorem 7.10 K is closed in the weak topology. Since K is bounded there exists $m \in \mathbb{N}$ with $K \subset mB_E$ and mB_E is weakly compact by Theorem 7.24. So K is a weakly closed subset of a weakly compact set and thus K is weakly compact. □

Corollary 7.33. *Let E be a reflexive Banach space and let $A \subset E$ be non-empty, closed and convex. Let $\varphi : A \to (-\infty, \infty]$ be a convex lower semi-continuous (l.s.c.) function such that $\varphi \not\equiv +\infty$ and*

$$\lim_{x \in A, \|x\| \to \infty} \varphi(x) = \infty \quad (\textit{no assumption if } A \textit{ is bounded}). \qquad (**)$$

Then φ achieves its minimum on A, i.e., there exists some $x_0 \in A$ such that

$$\varphi(x_0) = \inf_{x \in A} \varphi(x).$$

Proof. Fix any $a \in A$ such that $\varphi(a) < \infty$ and define

$$\tilde{A} := \{x \in A | \varphi(x) \leq \varphi(a)\}.$$

Then $\tilde{A}$ is closed, convex and bounded (by $(**)$) and thus compact in the weak topology $\sigma(E, E^*)$ by Corollary 7.32! By Corollary 7.13, φ is also l.s.c. in the weak topology $\sigma(E, E^*)$ (since φ is convex and strongly l.s.c).

Let $(x_n)_n \subset \tilde{A}$ be a minimizing sequence in $\tilde{A}$ (i.e., $x_n \in \tilde{A}, \varphi(x_n) \to \inf_{x \in \tilde{A}} \varphi(x)$). Since $\tilde{A}$ is weakly compact, $(x_n)_n$ has a weakly convergent subsequence, i.e.

$$x_0 := \text{weak} - \lim_{j \to \infty} x_{n_j} \text{ exists}$$

for some subsequence $(x_{n_j})_j$ of (x_n). Since $\tilde{A}$ is weakly closed it follows that $x_0 \in \tilde{A}$ and by the weak l.s.c. property of φ we get

$$\inf_{x \in \tilde{A}} \varphi(x) \leq \varphi(x_0) \leq \liminf_{l \to \infty} \varphi(x_{n_l}) = \inf_{x \in \tilde{A}} \varphi(x)$$

so $\varphi(x_0) = \inf_{x \in \tilde{A}} \varphi(x)$.
If $x \in A \setminus \tilde{A}$, then

$$\varphi(x_0) \leq \varphi(a) < \varphi(x),$$

thus $\varphi(x_0) < \varphi(x)\ \forall x \in A$. □

Remark 7.34. *Corollary 7.33 is the main reason why reflexive spaces and convex functions are so important in many problems in the calculus of variations.*

7.6 Separable spaces

Definition 7.35. *A metric space E is separable if there exists a countable dense subset $D \subset E$.*

Note: Many important spaces are separable. Finite-dimensional spaces are separable, also L^p and $l^p, 1 \leq p < \infty$ are separable. $C(K), K$ compact, is separable, but L^∞ and l^∞ are not separable.

Proposition 7.36. *Let E be a separable metric space and $F \subset E$ any subset. Then F is separable.*

Proof. Let $(u_n)_n \subset E$ be a countable dense subset of E and $r_m > 0, r_m \to \infty$ as $m \to \infty$. Choose any point $a_{m,n} \in B_{r_m}(u_n) \cap F$ whenever this is non-empty. Then $(a_{m,n})_{m,n}$ is countable and dense in F. □

Theorem 7.37. *Let E be a Banach space such that E^* is separable. Then E is separable.*

Remark 7.38. *The converse is not true! E.g., $E = L^1$ is separable, but $E^* = L^\infty$ is not.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be countable and dense in E^*. Since $\|f_n\| := \|f_n\|_{E^*} := \sup_{x\in E, \|x\|_E=1} |f_n(x)|$, there is some $x_n \in E$ such that

$$\|x_n\| = 1 \quad \text{and} \quad |f_n(x_n)| \geq \frac{1}{2}\|f_n\|. \tag{$*$}$$

Let L be the vector space over $\mathbb{F}$ generated by the $(x_n)_{n\in\mathbb{N}}$ (i.e., the set of finite linear combinations of the x_n).
Claim 1: L is dense in E.
Indeed, according to Remark 5.28 we have to check that any $f \in E^*$ which vanishes on L must be identically zero.
Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\|f - f_N\| < \varepsilon$. Then

$$\|f\| \leq \|f - f_N\| + \|f_N\|.$$

Note that since $f(x_N) = 0$ (f vanishes on L) and ($*$) we have

$$\frac{1}{2}\|f_N\| \leq \|f_N(x_N)\| = \|(f - f_N)(x_N)\| \leq \|f - f_N\|\|x_N\| = \|f - f_N\|.$$

So

$$\|f\| \leq \|f - f_N\| + 2\|f - f_N\| < 3\varepsilon$$

and since this holds for all $\varepsilon > 0$, $\|f\| = 0$, i.e., $f \equiv 0$.

If $\mathbb{F} = \mathbb{R}$, let L_0 be the vector space over $\mathbb{Q}$ generated by the $(x_n)_n$. If $\mathbb{F} = \mathbb{C}$ let L_0 be the vector space over $\mathbb{Q} + i\mathbb{Q}$ generated by the $(x_n)_n$. I.e., the set of all finite linear combinations with coefficients in $\mathbb{Q}$, resp. in $\mathbb{Q} + i\mathbb{Q}$.

Then L_0 is dense in L and hence dense in E (since L is dense in E by Claim 1).
Claim 2: L_0 is countable!
Indeed, for $n \in \mathbb{N}$ let Λ_n be the vector space over $\mathbb{Q}$, resp. over $\mathbb{Q}+i\mathbb{Q}$, generated by $(x_k)_{1\leq k\leq n}$. Λ_n is countable and

$$L_0 = \bigcup_{n\in\mathbb{N}} \Lambda_n$$

is countable, as a countable union of countable sets. □

Corollary 7.39. *Let E be a Banach space. Then E is reflexive and separable if and only if E^* is reflexive and separable.*

Proof. We already know by Theorem 7.37 and Corollary 7.31 that

$$E^* \text{reflexive and separable} \Rightarrow E \text{reflexive and separable.}$$

Conversely, if E is reflexive and separable, then $E^{**} = J(E)$ is reflexive and separable. Since $E^{**} = (E^*)^*$, the "$\Rightarrow$" direction applied to E^* yields E reflexive and separable. □

There is also a connection between separability and metrizability of the weak topologies.

Theorem 7.40. *Let E be a separable Banach space. Then B_{E^*} is metrizable in the weak$*$ topology $\sigma(E^*, E)$. Conversely, if B_{E^*} is metrizable in $\sigma(E^*, E)$, then E^* is separable.*

There is a dual statement.

Theorem 7.41. *Let E be a Banach space such that E^* is separable. Then B_E is metrizable in the weak topology $\sigma(E, E^*)$. Conversely, if B_E is metrizable in $\sigma(E, E^*)$, then E^* is separable.*

Proof of Theorem 7.40. Let $(x_n)_n \subset B_E$ be a dense countable subset of B_E. For $f \in E^*$ set

$$[f] := \sum_{n=1}^{\infty} \frac{1}{2^n} |f(x_n)|.$$

Then $[\cdot]$ is a norm on E^* and $[f] \leq \|f\|_{E^*}$ (Why?). Put $d(f, g) := [f - g]$. We have to show that the topology induced bu d on B_{E^*} is the same as the weak$*$ topology $\sigma(E^*, E)$ restricted to B_E.
Step 1: Let $f_0 \in B_{E^*}$ and V a neighborhood of f_0 in $\sigma(E^*, E)$. Have to find some $r > 0$ such that

$$U_r = \{f \in B_{E^*} | d(f, f_0) < r\} \subset V.$$

As before, we can assume that V is of the form

$$V = \{f \in B_{E^*} |\ |(f - f_0)(y_i)| < \varepsilon,\ \forall i = 1, \ldots, k\}$$

for some $\varepsilon > 0, y_1, \ldots, y_k \in E$.
W.l.o.g., $\|y_i\| \leq 1,\ i = 1, \ldots, k$.
Since $(x_n)_n$ is dense in B_E, we know that $\forall i = 1, \ldots, k, \exists n_i \in \mathbb{N}$ such that

$$\|y_i - x_{n_i}\| < \frac{\varepsilon}{4}.$$

Choose $r > 0$ small enough such that

$$2^{n_i} r < \frac{\varepsilon}{2}, \quad i = 1, \ldots, k.$$

Claim: $U_r \subset V$!
Indeed, if

$$r > d(f, f_0) = \sum_{n=1}^{\infty} \frac{1}{2^{n_i}} |(f - f_0)(x_{n_i})|$$

then

$$\frac{1}{2^{n_i}} |(f - f_0)(x_{n_i})| < r, \quad \forall i = 1, \dots, k.$$

Hence, for $i = 1, \dots, k$

$$\begin{aligned} |(f - f_0)(y_i)| &= |(f - f_0)(y_i - x_{n_i}) + (f - f_0)(x_{n_i})| \\ &\leq \underbrace{\|f - f_0\|}_{\leq 2} \underbrace{\|y_i - x_{n_i}\|}_{< \frac{\varepsilon}{4}} + \underbrace{|(f - f_0)(x_{n_i})|}_{< \frac{\varepsilon}{2}} < \varepsilon \end{aligned}$$

so $f \in V$.
Step 2: Let $f_0 \in B_{E^*}$. Given $r > 0$, we have to find some neighborhood V in $\sigma(E^*, E)$ such that

$$V \subset U = \{f \in B_{E^*} |\ d(f, f_0) < r\}.$$

We choose V to be of the form

$$V := \{f \in B_{E^*} |\ |(f - f_0)(x_i)| < \varepsilon\} \quad \forall i = 1, \dots, k$$

with ε and k to be determined so that $V \subset U$.
If $f \in V$, then

$$\begin{aligned} d(f, f_0) &= \sum_{n=1}^{\infty} \frac{1}{2^n} |(f - f_0)(x_n)| \\ &= \sum_{n=1}^{k} \frac{1}{2^n} \underbrace{|(f - f_0)(x_n)|}_{< \varepsilon} + \sum_{n=k+1}^{\infty} \frac{1}{2^n} \underbrace{|(f - f_0)(x_n)|}_{\leq 2} \\ &< \varepsilon + 2 \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \varepsilon + \frac{1}{2^{k-1}} \end{aligned}$$

so it is enough to take $\varepsilon = \frac{r}{2}$ and $k \in \mathbb{N}$ such that $\frac{1}{2^{k-1}} < \frac{r}{2}$.

Conversely, suppose that B_{E^*} is metrizable in $\sigma(E^*, E)$ and let us prove that E is separable.
Set

$$U_n := \{f \in B_{E^*} |\ d(f, 0) < \frac{1}{n}\}$$

and let V_n be a neighborhood of 0 in $\sigma(E^*, E)$ such that $V_n \subset U_n$. Again, we may assume that V_n has the form

$$V_n := \{f \in B_{E^*} |\ |f(x)| < \varepsilon_n\ \forall x \in \Phi_n\}$$