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Lectures Notes in Functinoal Analysis

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with $\varepsilon_n > 0$ and Φ_n some finite subset of E. Set

$$D := \bigcup_{n\in\mathbb{N}} \Phi_n$$

so that D is countable.
Claim: The vector space generated by D is dense in E (this implies E is separable!).
Suppose $f \in E^*$ is such that $f(x) = 0\ \forall x \in D$. Then $f \in V_n \subset U_n\ \forall n \in \mathbb{N}$. Thus $f \equiv 0$ (i.e., $span(D)$ is dense in E). □

"Proof of Theorem 7.41": The implication

$$E^* \text{ separable} \quad \Rightarrow \quad B_E \text{ is metrizable in } \sigma(E, E^*)$$

is exactly as above.
The proof of the converse is trickier (where does the above argument break down?). See Dunford-Schwartz: Linear Operators, Interscience, 1972. □

Corollary 7.42. *Let E be a Banach space and $(f_n)_n$ a bounded sequence in E^*. Then there exists a subsequence $(f_{n_l})_l$ that converges in the weak∗ topology $\sigma(E^*, E)$.*

Proof. W.l.o.g. $\|f_n\| \le 1\ \forall n \in \mathbb{N}$. The set B_{E^*} is compact (by Banach-Alaoglu) and metrizable (by Theorem 7.40) in the weak∗ topology $\sigma(E^*, E)$. So every sequence in B_{E^*} has a convergent subsequence! □

Proof of Theorem 7.28. Let $M_0 = span(x_n, n \in \mathbb{N})$ and $M = \overline{M_0}$. Clearly M is separable and $M \subset E$ is also reflexive (by Theorem 7.30). Thus $B_M =$ unit ball in M is compact and metrizable in the weak topology $\sigma(M, M^*)$, since M^* is separable (see Corollary 7.39 and Theorem 7.40). Hence there exists a subsequence $(x_{n_l})_l$ which converges weakly w.r.t. $\sigma(M, M^*)$ and hence $(x_{n_l})_l$ converges weakly w.r.t. $\sigma(E, E^*)$ also (see Proof of Theorem 7.30). □

7.7 Uniformly convex spaces

Definition 7.43. *A Banach space E is **uniformly convex** if $\forall\varepsilon > 0 \exists\delta > 0$ such that*

$$x, y \in E, \|x\| \le 1, \|y\| \le 1, \text{ and } \|x - y\| > \varepsilon \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

This is a geometric property of the unit ball. If one slides a ruler of length $\varepsilon > 0$ in the unit ball, then its midpoint must stay within a ball of radius $1 - \delta$ for some $\delta > 0$, i.e., it measures how round the unit sphere is.

Example. *(1) $E = \mathbb{R}^2, \|x\|_2 = (x_1^2 + x_2^2)^{\frac{1}{2}}$ is uniformly convex. Here the curvature of the unit sphere is positive.*
But

$$\|x\|_1 = |x_1| + |x_2| \quad (\textit{Manhattan norm})$$
$$\|x\|_\infty = \max(|x_1|, |x_2|)$$

are not uniformly convex. They both have a flat surface!

(2) L^p spaces are uniformly convex for $1 < p < \infty$. Any Hilbert space is uniformly convex.

Theorem 7.44. *[Milman-Pettis) Every uniformly convex Banach space is reflexive.*

Note:

- Uniform convexity is a **geometric property of the norm**, an equivalent norm need not be uniformly convex.
  Reflexivity is a **topological property**: a reflexive space remains reflexive for an equivalent norm.
  Thus Theorem 7.44 is somewhat surprising: a geometric property implies a topological property.

- Uniform convexity is often used to prove reflexivity, but this is only sufficient. There are (weird) reflexive Banach spaces that do not have any uniformly convex equaivalent norm!

Proof. Assume E is a real Banach space. Let $\xi \in E^{**}$, $\|\xi\| = 1$ and $J : E \to E^{**}$ be the canonical injection given by

$$J(x)(f) := f(x) \quad \forall f \in E^*, x \in E.$$

Have to show: $\xi \in J(B_E)$.
Since J is an isometry, $J(B_E) \subset E^{**}$ is closed in the strong topology on E^{**}. So it is enough to show

$$\forall \varepsilon > 0 \; \exists x \in B_E \text{ such that } \|\xi - J(x)\| \leq \varepsilon. \qquad (*)$$

Fix $\varepsilon > 0$ and let $\delta = \delta_\varepsilon > 0$ be the modulus of uniform convexity. Choose some $f \in E^*$ with $\|f\| = 1$ and

$$\xi(f) > 1 - \frac{\delta}{2} \quad \text{(if } E \text{ is real, otherwise work with } Re\xi(f)).$$

This is possible since $\|\xi\| = 1$.
Set

$$V := \{\eta \in E^{**} | \; |(\eta - \xi)(f)| < \frac{\delta}{2}$$

so $\xi \in V \in \sigma(E^{**}, E^*)$.
Since $J(B_E)$ is dense in $B_{E^{**}}$ w.r.t. weak$*$ topology $\sigma(E^{**}, E^*)$ thanks to Lemma 7.26 we have $V \cap J(B_E) \neq \emptyset$. Thus there is $x \in B_E$ such that $J(x) \in V$!
Claim: x satisfies $(*)$.
If not, then $\|\xi - J(x)\| > \varepsilon$, i.e.

$$\xi \in (J(x) + \varepsilon B_{E^{**}})^c := W \in \sigma(E^{**}, E^*) \quad (\text{since } B_{E^{**}} \text{ is closed in } \sigma(E^{**}, E^*)).$$

Then, again by Lemma 7.26, it follows $V \cap W \cap J(B_E) \neq \emptyset$, i.e.

$$\exists y \in B_E \text{ such that } J(y) \in V \cap W \subset V.$$

Note: Since $J(y) \in W$, we have $\|J(x) - J(y)\| \geq \varepsilon$, and since J is isometric, we must have

$$\|x - y\| > \varepsilon. \tag{$**$}$$

Since $J(x), J(y) \in V$ we have the inequalities

$$\begin{aligned}
\frac{\delta}{2} &> |(J(x) - \xi)(f)| = |f(x) - \xi(f)| \geq \xi(f) - f(x) \\
\frac{\delta}{2} &> |(J(y) - \xi)(f)| = |f(y) - \xi(f)| \geq \xi(f) - f(y)
\end{aligned}$$

$$\Rightarrow 2\xi(f) < f(x+y) + 2\delta \leq \|x+y\| + \delta$$

or

$$\Big\|\frac{x+y}{2}\Big\| > \xi(f) - \frac{\delta}{2} > 1 - \frac{\delta}{2} - \frac{\delta}{2} = 1 - \delta.$$

But by uniform convexity, this means

$$\|x - y\| < \varepsilon$$

contradicting ($**$). □

Proposition 7.45. *Let E be a uniformly convex Banach space and $(x_n)_n \subset E$ with $x_n \rightharpoonup x$ weakly in $\sigma(E, E^*)$ and*

$$\limsup \|x_n\| \leq \|x\|. \tag{I.20}$$

Then $x_n \to x$ strongly.

Remark. *We always have $x_n \rightharpoonup x \Rightarrow \|x\| \leq \liminf \|x_n\|$ (by Proposition 7.7), so (I.20) says that the sequence $\|x_n\|$ does not loose "mass" as $n \to \infty$.*

Proof. Assume $x \neq 0$ (otherwise trivial).
Idea: renormalize!
Set

$$\lambda_n := \max(\|x_n\|, \|x\|), \quad y_n := \frac{1}{\lambda_n} x_n, \quad y := \frac{x}{\|x\|}, \quad \text{so } \|y_n\| \leq 1, \|y\| = 1.$$

Note: $y_n \to y$ strongly implies $x_n \to x$ strongly (check this!).
Further note $\lambda_n \to \lambda$ and hence (since $x_n \rightharpoonup x$ weakly), $y_n \rightharpoonup y$ weakly (check this!). Thus

$$\frac{y_n + y}{2} \rightharpoonup y$$

and by Proposition 7.7

$$1 = \|y\| = \Big\|\frac{y+y}{2}\Big\| \leq \liminf \underbrace{\Big\|\frac{y_n + y}{2}\Big\|}_{\leq \frac{1}{2}(\|y_n\| + \|y\|) \leq 1}$$

$$\Rightarrow \Big\|\frac{y_n + y}{2}\Big\| \to 1 \quad \text{as } n \to \infty.$$

By the uniform convexity we get

$$\|y_n - y\| \to 0 \quad \text{as } n \to \infty,$$

i.e., $y_n \to y$ strongly. □

8 L^p spaces

Some notation: $(\Omega, \mathcal{A}, \mu)$ **measure space**, i.e., Ω is a set and

(i) $\mathcal{A}$ is a σ-algebra in Ω: a collection of subsets of Ω (so $\mathcal{A} \subset \mathcal{P}(\Omega)$) such that

  (a) $\emptyset \in \mathcal{A}$

  (b) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

  (c) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ whenever $A_n \in \mathcal{A}\ \forall n \in \mathbb{N}$

(ii) μ is a measure, i.e., $\mu : \mathcal{A} \to [0, \infty]$ with

  (a) $\mu(\emptyset) = 0$

  (b) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} \mu(A_n)$ whenever $(A_n)_n \subset \mathcal{A}$ are disjoint

We will also assume that

(iii) Ω is σ-finite, i.e., there exist $\Omega_n \in \mathcal{A}, n \in \mathbb{N}$ which exhaust Ω, i.e., $\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega$, and $\mu(\Omega_n) < \infty\ \forall n \in \mathbb{N}$.

The sets $N \in \mathcal{A}$ such that $\mu(N) = 0$ are called null sets.

A property holds almost everywhere (a.e.) or for almost all $x \in \Omega$, if it holds everywhere on $\Omega \setminus N$, where N is a null set.

See Bauer: Measure theory, 4th edition, and the handout for details on **measurable functions** $f : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$).

We denote by $L^1(\Omega, \mu)$ (or simply $L^1(\Omega)$, or just L^1) the space of integrable function from Ω to $\mathbb{R}/\mathbb{C}$.

We often write $\int f = \int f d\mu = \int\limits_{\Omega} f d\mu$,

$$\|f\|_1 = \|f\|_{L^1} = \int\limits_{\Omega} |f| d\mu = \int |f|.$$

As usual, we identify functions which coincide a.e.!

8.1 Some results from integration everyone must know

Theorem (Monotone convergence, Beppo-Levi). *Let $(f_n)_n$ be a sequence of non-negative functions in L^1 which is increasing,*

$$f_1 \le f_2 \le \cdots \le f_n \le f_{n+1} \le \ldots \text{ a.e. on } \Omega,$$

and bounded, $\sup_{n \in \mathbb{N}} \int f_n d\mu < \infty$. *Then*

$$f(x) := \lim_{n \to \infty} f_n(x)$$

exists a.e., $f \in L^1$, and $\|f - f_n\|_{L^1} \to 0$.

Theorem (Dominated convergence, Lebesgue). *Let $(f_n)_n \subset L^1$ be such that*

(a) $f_n(x) \to f(x)$ a.e. on Ω

(b) there exists $g \in L^1$ such that for all $n \in \mathbb{N}$

$$|f_n(x)| \leq g(x) \quad a.e.$$

Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.

Lemma (Fatou). *Let $(f_n)_n \subset L^1$ with*

(a) $\forall n \in \mathbb{N} : f_n(x) \geq 0$ a.e.

(b) $\sup_{n\in\mathbb{N}} \int f_n d\mu < \infty$

Set $f(x) := \liminf_{n\to\infty} f_n(x) \leq \infty$. Then $f \in L^1$ and

$$\int f d\mu \leq \liminf_{n\in\mathbb{N}} \int f_n d\mu.$$

Basic example: $\Omega = \mathbb{R}^d, \mathcal{A} =$ Borel-measurable sets (or Lebesgue-measurable sets) and $\mu =$ Lebesgue measure on $\mathbb{R}^d$.

Notation: $C_c(\mathbb{R}^d) =$ space of continuous functions on $\mathbb{R}^d$ with compact support, i.e.,

$$C_c(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) | \exists K \subset \mathbb{R}^d \text{ compact such that } f(x) = 0 \; \forall x \in K^c\}.$$

Theorem (Density). *$C_c(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$, i.e., $\forall f \in L^1(\mathbb{R}^d) \forall \varepsilon > 0 \; \exists g \in C_c(\mathbb{R}^d)$ with $\|f - g\|_1 < \varepsilon$.*

The case of product measures (and spaces): $(\Omega, \mathcal{A}_1, \mu_1), (\Omega, \mathcal{A}_2, \mu_2)$ two σ-finite measure spaces

$$\begin{aligned}
&\Omega := \Omega_1 \times \Omega_2 \\
&\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \\
&\mu = \mu_1 \otimes \mu_2 \quad \text{by } \mu(A_1 \times A_2) := \mu_1(A_1) \cdot \mu_2(A_2) \; \forall A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2.
\end{aligned}$$

Theorem (Tonelli). *Let $F(= F(x, y)) : \Omega_1 \times \Omega_2 \to [0, \infty]$ be measurable and*

(a) $\int\limits_{\Omega_2} F(x, y) d\mu_2 < \infty$ a.e. in Ω_1,

(b) $\int\limits_{\Omega_1} \Big(\int\limits_{\Omega_2} F(x, y) d\mu_2 \Big) d\mu_1 < \infty$.

Then $F \in L^1(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ and

$$\begin{aligned}
\int\limits_{\Omega_1} \Big(\int\limits_{\Omega_2} F(x, y) d\mu_2 \Big) d\mu_1 &= \int\limits_{\Omega_2} \Big(\int\limits_{\Omega_1} F(x, y) d\mu_1 \Big) d\mu_2 \\
&= \int \int\limits_{\Omega_1 \times \Omega_2} F(x, y) d(\mu_1 \otimes \mu_2).
\end{aligned}$$

Theorem (Fubini). *If $F \in L^1(\Omega_1 \times \Omega_2)$, i.e.,*

$$\int\limits_{\Omega_1 \times \Omega_2} |F(x, y)| d(\mu_1 \otimes \mu_2) < \infty,$$

then

(a) for a.e. $x \in \Omega_1 : F(x,\cdot) \in L^1(\Omega_2)$ *and* $\int\limits_{\Omega_2} F(x,y)d\mu_2 \in L^1_x(\Omega_1)$

(b) for a.e. $y \in \Omega_2 : F(\cdot,y) \in L^1(\Omega_1)$ *and* $\int\limits_{\Omega_1} F(x,y)d\mu_1 \in L^1_y(\Omega_2)$

Moreover

$$\int\limits_{\Omega_1}\Big(\int\limits_{\Omega_2} F(x,y)d\mu_2\Big)d\mu_1 = \int\limits_{\Omega_2}\Big(\int\limits_{\Omega_1} F(x,y)d\mu_1\Big)d\mu_2 = \int\int\limits_{\Omega_1\times\Omega_2} F(x,y)d\mu_1 d\mu_2.$$

8.2 Definition and some properties of L^p spaces

Definition 8.1. • $1 \le p < \infty$:

$$L^p = L^p(\Omega,\mathbb{F}) := \{f : \Omega \to \mathbb{F} | f \textit{ is measurable and } |f|^p \in L^1\},$$

$$\|f\|_p := \|f\|_{L^p} := \Big(\int\limits_{\Omega} |f(x)|^p d\mu\Big)^{\frac{1}{p}}.$$

• $p = \infty$:

$$L^\infty = L^\infty(\Omega,\mathbb{F}) := \{f : \Omega \to \mathbb{F} | f \textit{ is measurable and there exists a constant } C < \infty \textit{ such that } |f(x)| \le C a.e. \textit{ on } \Omega\},$$

$$\|f\|_\infty := \|f\|_{L^\infty} := \inf(C|\ |f(x)| \le C a.e. \textit{ on } \Omega\} =: esssup_{x\in\Omega}|f(x)|.$$

Remark. *If* $f \in L^\infty$ *then*

$$|f(x)| \le \|f\|_\infty \quad a.e. \textit{ on } \Omega.$$

Indeed, by definition of $\|f\|_\infty$*, there exists* $C_n \searrow \|f\|_\infty$ *(e.g.* $C_n = \|f\|_\infty + \frac{1}{n}$*) such that*

$$|f(x)| \le C_n \quad a.e. \textit{ on } \Omega,$$

i.e., $\exists N_n$ *such that* $|f(x)| \le C_n\ \forall x \in \Omega \setminus N_n$ *and* $\mu(N_n) = 0$.
Set $N := \bigcup_n N_n$ *and note*

$$\mu(N) \le \sum_{n\in\mathbb{N}} \mu(N_n) = 0$$

and for all $n \in \mathbb{N}$:

$$|f(x)| \le C_n \quad \forall x \in \Omega \setminus N$$

$$\Rightarrow |f(x)| \le \|f\|_\infty \quad \forall x \in \Omega \setminus N.$$

Notation: If $1 \le p \le \infty$, then p' given by $\frac{1}{p} + \frac{1}{p'} = 1$ is the **dual exponent** of p.

Theorem 8.2 (Hölder). *Let $f \in L^p$ and $g \in L^{p'}$ with $1 \le p \le \infty$. Then $fg \in L^1$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_{p'}.$$

Proof. Obvious for $p = 1$ or $p = \infty$.
So assume $1 < p < \infty$, and note that for all $a, b \ge 0$

$$\begin{aligned} ab &= \frac{1}{p}a^p + ab - \frac{1}{p}a^p \\ &\le \frac{1}{p}a^p + \sup_{b \ge 0}(ab - \frac{1}{p}a^p) \\ &= \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \quad (\text{ also called Young's inequality}) \end{aligned}$$

Thus

$$\begin{aligned} |f(x)g(x)| &\le \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{p'} \quad \text{a.e.} \\ &\in L^1 \text{ since } f \in L^p, g \in L^{p'}. \end{aligned}$$

Moreover,

$$\int |fg| d\mu \le \frac{1}{p}\|f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'}.$$

So for $\lambda > 0$,

$$\begin{aligned} \int |fg| d\mu = \int |\lambda f \frac{1}{\lambda} g| d\mu &\le \frac{1}{p}\|\lambda f\|_p^p + \frac{1}{p'}\|\lambda^{-1} g\|_{p'}^{p'} \\ &= \frac{\lambda^p}{p}\|f\|_p^p + \frac{\lambda^{-p'}}{p'}\|g\|_{p'}^{p'} = h(\lambda). \end{aligned}$$

Minimizing over $\lambda > 0$ yields the claim, since

$$\inf_{\lambda > 0} h(\lambda) = \|f\|_p \|g\|_{p'} \quad (\text{check this!})$$

$\square$

Remark. *There is a very useful extension of Hölder in the form: If $f_1, f_2, \ldots, f_k$ are such that $f_j \in L^{p_j}$ for $1 \le j \le k$ and $\frac{1}{p} = \sum_{j=1}^{k} \frac{1}{p_j}$, then $f = f_1 \cdot f_2 \cdot \ldots f_k \in L^p$ and*

$$\|f\|_p \le \prod_{j=1}^{k} \|f_j\|_{p_j}.$$

In particular, if $f \in L^p \cap L^q$ for some $1 \le p \le q \le \infty$, then $f \in L^r$ for all $p \le r \le q$ and

$$\|f\|_r \le \|f\|_p^\theta \|g\|_q^{1-\theta} \quad \textit{with } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},\ 0 \le \theta \le 1.$$

Theorem 8.3. *L^p is a vector space and $\|\cdot\|_p$ is a norm for any $1 \le p \le \infty$.*

Proof. The cases $p = 1$ and $p = \infty$ are easy, so assume $1 < p < \infty$.
If $f, f \in L^p$, then

$$\begin{aligned} |f+g|^p &\le (|f|+|g|)^p \le (2\max(|f|,|g|))^p \\ &= 2^p \max(|f|^p, |g|^p) \le 2^p(|f|^p + |g|^p) \in L^p. \end{aligned}$$

Moreover,

$$\begin{aligned} \|f+g\|_p^p &= \int |f+g|^{p-1}|f+g| d\mu \\ &\le \int |f+g|^{p-1}|f| d\mu + \int |f+g|^{p-1}|g| d\mu. \end{aligned} \tag{$*$}$$

Note that $p' = \frac{p}{p-1}$, so

$$\left(|f+g|^{p-1}\right)^{p'} = |f+g|^p \in L^1$$

so $|f+g|^{p-1} \in L^{p'}$ and by Hölder, ($*$) yields

$$\|f+g\|_p^p \le \||f+g|^{p-1}\|_{p'}(\|f\|_p + \|g\|_p) = \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p).$$

Since $\|f+g\|_p \le \infty$, this yields

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

□

Theorem 8.4 (Fischer-Riesz). *L^p is a Banach space for $1 \le p \le \infty$.*

Proof. We distinguish the cases $p = \infty$ and $1 \le p < \infty$.
Case 1: $p = \infty$: Let $(f_n)_n \subset L^\infty$ be Cauchy. Given $k \in \mathbb{N}$ $\exists N_k \in \mathbb{N}$ such that $\|f_m - f_n\|_\infty \le \frac{1}{k}$ for $m, n \ge N_k$. Hence there exists a set $E_k \subset \Omega, \mu(E_k) = 0$, such that

$$|f_m(x) - f_n(x)| \le \frac{1}{k} \quad \forall x \in \Omega \setminus E_k \text{ and all } m, n \ge N_k.$$

Put $E := \bigcup_{k \in \mathbb{N}} E_k$, note $\mu(E) = 0$ and

$$\forall x \in \Omega \setminus E: \quad |f_m(x) - f_n(x)| \le \frac{1}{k} \quad \text{for all } m, n \ge N_k, \tag{$*$}$$

that is, the sequence $(f_n(x))_n$ is Cauchy (in $\mathbb{R}$). So

$$f(x) := \lim_{n \to \infty} f_n(x)$$

exists for all $x \in \Omega \setminus E$ and we simply set $f(x) := 0$ for $x \in E$.
Letting $m \to \infty$ in ($*$), we also see

$$|f(x) - f_n(x)| \le \frac{1}{k} \quad \forall x \in \Omega \setminus E \text{ and all } n \ge N_k.$$

So

$$|f(x)| \leq \underbrace{|f(x) - f_n(x)|}_{\leq \frac{1}{k}} + \underbrace{f_n(x)}_{\leq \|f_n\|_\infty} \quad \text{for a.a. } x \in \Omega.$$

Hence $f \in L^\infty$ and $\|f - f_n\|_\infty \leq \frac{1}{k}$ for all $n \geq N_k$. Thus $f_n \to f$ in L^∞!

Case 2: $1 \leq p < \infty$:

Step 1: Let $(f_n)_n \subset L^p$ be Cauchy. It is enough to show that there is a subsequence $(f_{n_l})_l$ that converges to some $f \in L^p$. Indeed, assume that $f_{n_l} \to f$ in L^p. Then

$$\|f - f_m\|_p \leq \|f - f_{n_l}\|_p + \|f_{n_l} - f_m\|_p,$$

so if $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$\|f - f_{n_l}\|_p < \frac{\varepsilon}{2} \quad \forall l \geq N_1$$

and there exists $N_2 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < \frac{\varepsilon}{2} \quad \forall m, n \geq N_2.$$

Note that $n_l \geq n$ (because of subsequence) so

$$\|f_{n_l} - f_m\|_p < \frac{\varepsilon}{2} \quad \forall l, m \geq N_2.$$

Hence for $l \geq \max(N_1, N_2)$ one has

$$\|f - f_m\|_p \leq \|f - f_{n_l}\|_p + \|f_{n_l} - f_m\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall m \geq N_1,$$

i.e., $f_, \to f$ in L^p.

Step 2: There exists a subsequence (f_{n_l}) which converges in L^p.

Extract a subsequence (f_{n_l}) such that

$$\|f_{n_{l+1}} - f_{n_l}\|_p \leq \frac{1}{2^l} \quad \forall l \in \mathbb{N}.$$

(To see that this exists proceed inductively: Choose $n_1 \in \mathbb{N}$ such that $\|f_m - f_n\|_p < \frac{1}{2}\ \forall m, n \geq n_1$. Then choose $n_2 \geq n_1$ such that $\|f_m - f_n\|_p < \frac{1}{2^2}\ \forall m, n \geq n_2$, etc.).

Claim: f_{n_l} converges to some f in L^p. Indeed, writing f_l instead of f_{n_l}, we have

$$\|f_{l+1} - f_l\|_p < \frac{1}{2^l} \quad \forall l \in \mathbb{N}.$$

Set

$$g_n(x) := \sum_{l=1}^{n} |f_{l+1}(x) - f_l(x)|$$

and note that the sequence $(g_n)_n$ is increasing. Also note that

$$\|g_n\|_p \leq \sum_{l=1}^{n} \|f_{l+1} - f_l\|_p < \sum_{l=1}^{n} \frac{1}{2^l} < \sum_{l=1}^{\infty} \frac{1}{2^l} = 1.$$