جامعة الانبار

كلية التربية للعلوم الصرفة

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Lectures Notes in Functinoal Analysis

WS 2012 – 2013

So

$$\sup_n \|g_n\|_p \le 1$$

and hence, by monotone convergence, $g_n(x)$ converges to a finite limit, say

$$g(x) = \lim_{n\to\infty} g_n(x) = \sup_n g_n(x) \quad \text{for a.a. } x.$$

If $m, n \ge 2$, then

$$\begin{aligned} |f_m(x) - f_n(x)| &\le |f_m(x) - f_{m-1}(x)| + \cdots + |f_{n+1}(x) - f_n(x)| \\ &\le g(x) - g_{n-1}(x) \to 0 \quad \text{a.e.} \end{aligned}$$

So for a.e. x, $(f_n(x))_n$ is Cauchy and converges to some finite limit, denoted by $f(x)$, say. Letting $m \to \infty$, we also see, for a.e. x,

$$|f(x) - f_n(x)| \le g(x) - g_{n-1}(x) \le g(x) \quad \text{for } n \ge 2.$$

In particular, $f \in L^p$ and, since $g^p \in L^1$ and $f(x) - f_n(x) \to 0$ a.e. as $n \to \infty$, we can also apply dominated convergence to see

$$\|f - f_n\|_p \to 0 \quad \text{as } n \to \infty.$$

□

8.3 Reflexivity, Separability. The Dual of L^p

We will consider the three cases

(A) $1 < p < \infty$

(B) $p = 1$

(C) $p = \infty$

(A) Study of L^p for $1 < p < \infty$.

This is the most favorable case: L^p is reflexive, separable, and the dual of L^p is $L^{p'}$.

Theorem 8.5. *L^p is reflexive for $1 < p < \infty$.*

Proof. Step 1: (Clarkson's first inequality) Let $2 \le p < \infty$. Then

$$\Big\|\frac{f+g}{2}\Big\|_p^p + \Big\|\frac{f-g}{2}\Big\|_p^p \le \frac{1}{2}(\|f\|_p^p + \|g\|_p^p) \quad \forall f, g \in L^p. \tag{1}$$

Proof of (1). Enough to show

$$\Big|\frac{a+b}{2}\Big|^p + \Big|\frac{a-b}{2}\Big|^p \le \frac{1}{2}(|a|^p + |b|^p) \quad \forall a, b \in \mathbb{R}.$$

Note that

$$\alpha^p + \beta^p \le (\alpha^2 + \beta^2)^{\frac{p}{2}} \quad \forall \alpha, \beta \ge 0. \tag{2}$$

Indeed, if $\beta > 0$, then (2) is equivalent to

$$\left(\frac{\alpha}{\beta}\right)^p + 1 \leq \left(\left(\frac{\alpha}{\beta}\right)^2 + 1\right)^{\frac{p}{2}} \tag{3}$$

and the function $(x^2+1)^{\frac{p}{2}} - x^p - 1$ increases on $[0,\infty)$ and equals 0 at $x = 0$, so

$$(x^2+1)^{\frac{p}{2}} - x^p - 1 \geq 0 \quad \forall x \geq 0.$$

Hence (3) and thus (2) hold.

Now choose $\alpha = \left|\frac{a+b}{2}\right|, \beta = \left|\frac{a-b}{2}\right|$ in (2) to see

$$\begin{aligned}\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p &\leq \left(\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2\right)^{\frac{p}{2}} \\ &= \left(\frac{a^2+b^2}{2}\right)^{\frac{p}{2}} \leq \frac{1}{2}(a^p + b^p),\end{aligned}$$

where in the last inequality we used the convexity of the function $x \mapsto x^{\frac{p}{2}}$ for $p \geq 2$.

□

Step 2: L^p is uniformly convex, and thus reflexive, for $2 \leq p < \infty$.
Indeed, let $f, g \in L^p, \|f\|_p \leq 1, \|g\|_p \leq 1$ and $\|f - g\| \geq \varepsilon$. Then from (1) we get

$$\left|\left|\frac{f+g}{2}\right|\right|_p^p \leq \frac{1}{2}(\|f\|_p^p + \|g\|_p^p) - \left|\left|\frac{f-g}{2}\right|\right|_p^p \leq 1 - \left(\frac{\varepsilon}{2}\right)^p$$

$$\Rightarrow \left|\left|\frac{f+g}{2}\right|\right|_p \leq \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}} = 1 - \underbrace{\left(1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}\right)}_{=\delta_\varepsilon > 0}.$$

So $L^p, 2 \leq p < \infty$, is uniformly convex and hence reflexive by Theorem 7.44.
Step 3: L^p is reflexive for $1 < p \leq 2$.
Indeed, let $1 < p < \infty$ and consider $T : L^p \to (L^{p'})^*, \frac{1}{p} + \frac{1}{p'} = 1$, defined as follows: given $u \in L^p$, the mapping

$$L^{p'} \ni f \mapsto \int uf d\mu$$

is a continuous linear functional on $L^{p'}$ (by Hölder) and thus defines an element $Tu \in (L^{p'})^*$ such that

$$(Tu)(f) = \int uf d\mu \quad \forall f \in L^{p'}.$$

Claim:

$$\|Tu\|_{(L^{p'})^*} = \|u\|_{L^p} \quad \forall u \in L^p.$$

Proof. By Hölder

$$|Tu(f)| = |\int ufd\mu| \leq \int |u||f|d\mu \leq \|u\|_p\|f\|_{p'} \quad \forall f \in L^{p'}$$

so

$$\|Tu\|_{(L^{p'})^*} = \sup_{\|f\|_p=1} |\int ufd\mu| \leq \|u\|_p.$$

On the other hand, given $u \in L^p$, we set

$$f_0(x) := \begin{cases} \lambda|u(x)|^{p-2}\overline{u(x)}, & \text{if } u(x) \neq 0 \\ 0, & \text{else} \end{cases}$$

and note that, since $p' = \frac{p}{p-1}$,

$$\int |f_0(x)|^{p'} d\mu = \lambda^{p'} \int (|u|^{p-1})^{p'} d\mu = \lambda^{p'} \int |u|^p d\mu = \lambda^{p'}\|u\|_p^p$$

so

$$\|f_0\|_{p'} = \lambda\|u\|_p^{p-1} = 1 \quad \text{if } \lambda = \frac{1}{\|u\|_p^{p-1}}.$$

With this choice of f, we have

$$\|Tu\|_{(L^{p'})^*} \geq |Tu(f_0)| = \|u\|_p$$

so the claim follows and $T : L^p \to (L^{p'})^*$ is an isometry!. Since L^p is a Banach space, we see that $T(L^p)$ is a closed subspace of $(L^{p'})^*$.

Now assume $1 < p \leq 2$. Since $2 < p' < \infty$, we know from Step 2, that $L^{p'}$ is reflexive. Since a Banach space E is reflexive if and only if its dual E^* is reflexive, we see that $(L^{p'})^*$ is also reflexive and since every closed subspace of a reflexive space is also reflexive, we see that $T(L^p)$ is reflexive and thus L^p too. □

□

Remark. *L^p is also uniformly convex for $1 < p \leq 2$ due to Clarkson's second inequality*

$$\Big|\Big|\frac{f+g}{2}\Big|\Big|_p^{p'} + \Big|\Big|\frac{f-g}{2}\Big|\Big|_p^{p'} \leq \Big(\frac{1}{2}(\|f\|_p^p + \|g\|_p^p)\Big)^{\frac{1}{p-1}}$$

which is trickier to prove than his first inequality.

Theorem 8.6 (Riesz representation theorem)**.** *Let $1 < p < \infty$ and $\phi \in (L^p)^*$. Then there exists a unique $u \in L^{p'}$ such that*

$$\phi(f) = \int ufd\mu.$$

Moreover,

$$\|u\|_{p'} = \|\phi\|_{(L^p)^*}.$$

Remark. *Theorem 8.6 is extremely important! It says that every continuous linear functional on L^p with $1 < p < \infty$ can be represented in a "concrete way" as an integral. The mapping $\phi \mapsto u$ is linear and surjective and allows us to identify the abstract space $(L^p)^*$ with $L^{p'}$! It is the sole reason why one always makes identification $(L^p)^* = L^{p'}$ for $1 < p < \infty$.*

Proof. Consider $T : L^{p'} \to (L^p)^*$ defined by

$$Tu(f) := \int uf d\mu \quad \forall u \in L^{p'}, f \in L^p$$

and note that by Step 3 in the proof of Theorem 8.5 one has

$$\|Tu\|_{(L^p)^*} = \|u\|_{p'} \quad \forall u \in L^{p'}.$$

So we only have to check that T is surjective. Indeed, let $E = T(L^{p'})$ which is a closed subspace of $(L^p)^*$. So it is enough to show that E is dense in $(L^p)^*$. For this, let $h \in (L^p)^{**}$ satisfy

$$h(\phi) = 0 \quad \forall \phi \in E,$$

i.e., $h(Tu) = 0 \; \forall u \in L^{p'}$. Since L^p is reflexive, $h \in L^p$ and

$$h(Tu) = Tu(h) = \int uh d\mu.$$

So we have

$$\int uh d\mu = 0 \quad \forall u \in L^{p'}.$$

Choosing

$$u = |h|^{p-2}\bar{h} \in L^{p'}$$

one sees

$$0 = \int uh d\mu = \int |h|^p d\mu$$

so $h = 0$. Hence every continuous linear functional on $E \subset (L^p)^*$ vanishes on $(L^p)^*$, so E is dense in $(L^p)^*$. □

Theorem 8.7. *The space $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for every $1 \le p < \infty$.*

Some notations:

- Truncation operator $T_n : \mathbb{R} \to \mathbb{R}$,

$$T_n(r) := \begin{cases} r, & \text{if } |r| \le n, \\ \frac{nr}{|r|}, & \text{if } |r| > n. \end{cases}$$

- Characteristic function: for $E \subset \Omega$ let

$$\mathbb{1}_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{else.} \end{cases}$$

Proof of Theorem 8.7. Step 1: $L^p \cap L_c^\infty$ is dense in L^p. (L_c^∞ = bounded functions with compact support).
Indeed, let $f \in L^p$. Put

$$g_n := \mathbb{1}_{B_n} T_n(f) \in L_c^\infty,$$

where $B_n = B_n(0) = \{x \in \mathbb{R}^d | \, |x| < n\}$. Since $|g_n| \le |f| \in L^p \; \forall n$ and $g_n \to f$ a.e., Dominated convergence yields

$$\|g_n - f\|_p \to 0 \quad \text{as } n \to \infty.$$

Step 2: $C_c(\mathbb{R}^d)$ is dense in $L^p \cap L_c^\infty$ w.r.t. $\|\cdot\|_p$.
Indeed, let $f \in L^p \cap L_c^\infty$. Since f is bounded and has compact support, we have $f \in L^1$ also. Let $\varepsilon > 0$. By density of $C_c(\mathbb{R}^d)$ in L^1, for any $\delta > 0$ there exists $g \in C_c(\mathbb{R}^d)$ such that

$$\|f - g\|_1 < \delta.$$

W.l.o.g., we may assume that $\|g\|_\infty \le \|f\|_\infty$, otherwise simply replace g by $T_n(g)$ with $n = \|f\|_\infty$.
Now note

$$\|f - g\|_p \le \|f - g\|_1^{\frac{1}{p}} \|f - g\|_\infty^{1-\frac{1}{p}} \le \delta^{\frac{1}{p}} (2\|f\|_\infty)^{1-\frac{1}{p}}.$$

Choosing δ so small that $\delta^{\frac{1}{p}} (2\|f\|_\infty)^{1-\frac{1}{p}} < \varepsilon$ we see

$$\|f - g\|_p < \varepsilon.$$

□

Theorem 8.8. *$L^p(\mathbb{R}^d)$ is separable for any $1 \le p < \infty$.*

Remark. *As a consequence, $L^p(\Omega)$ is separable for any measurable subset $\Omega \subset \mathbb{R}^d$. Indeed, let I be the canonical isometry from $L^p(\Omega)$ into $L^p(\mathbb{R}^d)$ by extending a function $f : \Omega \to \mathbb{F}$ to $\mathbb{R}^d$ by setting it zero outside Ω. Then $L^p(\Omega)$ may be identified with a subspace of $L^p(\mathbb{R}^d)$, hence $L^p(\Omega)$ is also separable, whenever $L^p(\mathbb{R}^d)$ is! (see Theorem 7.36).*

Proof of Theorem 8.8. Let $\mathcal{R}$ be the countable family of sets of the form

$$R = \prod_{k=1}^{d} (a_k, b_k), \quad a_k, b_k \in \mathbb{Q}$$

and $\mathcal{E}$ = vector space over $\mathbb{Q}$ (or $\mathbb{Q} + i\mathbb{Q}$) generated by the functions $(\mathbb{1}_R)_{R \in \mathcal{R}}$. So $\mathcal{E}$ is countable, since $\mathcal{E}$ consists of finite linear combinations with rational coefficients of functions $\mathbb{1}_R$.
Claim: $\mathcal{E}$ is dense in $L^p(\mathbb{R}^d)$.
Indeed, given $f \in L^p(\mathbb{R}^d), \varepsilon > 0 \; \exists f_1 \in C_c(\mathbb{R}^d)$ such that $\|f - f_1\|_p < \frac{\varepsilon}{2}$.
Let $R \in \mathcal{R}$ be any cube such that $supp(f) \subset R$.
Subclaim: Given any $\delta > 0$, there exists a function $f_2 \in \mathcal{E}$ such that $\|f_1 - f_2\|_p < \delta$ and $supp(f_2) \subset R$.

Indeed, simply split R into small cubes in $\mathcal{R}$ where the oscillation $(\sup - \inf)$ of f_1 is less than δ. Then

$$\|f_1 - f_2\|_p \leq \|f_1 - f_2\|_\infty |R|^{\frac{1}{p}} < \delta |R|^{\frac{1}{p}},$$

where $|R| =$ volume of R. By choosing $\delta > 0$ such that $\delta|R|^{\frac{1}{p}} < \frac{\varepsilon}{2}$ we have

$$\|f - f_2\|_p \leq \|f - f_1\|_p + \|f_1 - f_2\|_p < \varepsilon$$

and $f_2 \in \mathcal{E}$. □

(B) Study of L^1.
The dual space to L^1 is described in

Theorem 8.9 (Riesz representation theorem). *Let $\phi \in (L^1)^*$. Then there exists a unique function $u \in L^\infty$ such that*

$$\phi(f) = \int u f d\mu \quad \forall f \in L^1.$$

Moreover

$$\|u\|_\infty = \|\phi\|_{(L^1)^*}.$$

Remark. *Again, Theorem 8.9 allows us to identify every abstract continuous linear functional $\phi \in (L^1)^*$ with a concrete integral. The mapping $\phi \mapsto u$, which is a linear surjective isometry allows to identify the abstract space $(L^1)^*$ with L^∞. Therefore, one usually makes the identification $(L^1)^* = L^\infty$.*

Proof. Recall that we assume that Ω is σ-finite, i.e., there exists a sequence $\Omega_n \subset \Omega$ of measurable sets such that $\Omega = \bigcup_n \Omega_n$ and $\mu(\Omega_n) < \infty \ \forall n$. Set $\chi_n := \mathbb{1}_{\Omega_n}$.
Uniqueness of u: Suppose $u_1, u_2 \in L^\infty$ satisfy

$$\phi(f) = \int u_1 f d\mu = \int u_2 f d\mu \quad \forall f \in L^1.$$

Then $u = u_1 - u_2$ satisfies

$$\int u f d\mu = 0 \quad \forall f \in L^1. \qquad (*)$$

Let

$$sign\ u = \begin{cases} \frac{\bar{u}}{|u|^2}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0, \end{cases}$$

and choose $f = \mathbb{1}_n sign\ u$ in $(*)$. Then

$$\int_{\Omega_n} |u| d\mu = 0 \quad \forall n$$

so $u = 0$ on Ω_n, hence $u = 0$.
Existence of u: Step 1: There is a function $\theta \in L^2$ such that

$$\theta(x) \geq \varepsilon_n > 0 \quad \forall x \in \Omega_n \; \forall n.$$

Indeed, let $\theta = \alpha_1$ on Ω_1, $\theta = \alpha_2$ on $\Omega_2 \setminus \Omega_1$, $\ldots, \theta = \alpha_n$ on $\Omega_n \setminus \Omega_{n-1}$, etc. and adjust the constants $\alpha_n > 0$ so that $\theta \in L^2$.
Step 2: Given $\theta \in (L^1)^*$, the mapping

$$L^2 \ni f \mapsto \phi(\theta f)$$

defines a continuous linear functional on L^2! So by the Riesz representation theorem for L^2, there exists a function $v \in L^2$ such that

$$\phi(\theta f) = \int v f d\mu \quad \forall f \in L^2. \tag{$**$}$$

Set $u(x) := \frac{v(x)}{\theta(x)}$ (well-defined since $\theta > 0$ on Ω). Note that u is measurable and, with $\chi_n := \mathbb{1}_{\Omega_n}$, we have $u\chi_n \in L^2 \; \forall n$.
Claim: u has all the required properties.
Choosing $f = \chi_n \frac{g}{\theta} \in L^2$ for $g \in L^\infty$ in ($**$) we have

$$\phi(\chi_n g) = \int u \chi_n g d\mu \quad \forall g \in L^\infty. \tag{$***$}$$

Claim: $u \in L^\infty$ and $\|u\|_\infty \leq \|\phi\|_{(L^1)^*}$.

Proof. Fix $C > \|\phi\|_{(L^1)^*}$ and set

$$A := \{x \in \Omega |\; |u(x)| > C\}.$$

We need to show that $\mu(A) = 0$.
Choosing $g = \chi_A sign\; u$ in ($***$), we see

$$\begin{aligned} \int_{A \cap \Omega_n} |u| d\mu = \int u \chi_n g d\mu &= \phi(\chi_n g) \\ &\leq \|\phi\|_{(L^1)^*} \|\chi_n g\|_1 \\ &= \|\phi\|_{(L^1)^*} \mu(A \cap \Omega_n). \end{aligned}$$

Note that $|u| > C$ on A, so

$$\int_{A \cap \Omega_n} |u| d\mu \geq C \int_{A \cap \Omega_n} d\mu = C\mu(A \cap \Omega_n)$$

and thus

$$C\mu(A \cap \Omega_n) \leq \|\phi\|_{(L^1)^*} \mu(A \cap \Omega_n),$$

so, since $C > \|\phi\|_{(L^1)^*}$, we must have

$$\mu(A \cap \Omega_n) = 0 \quad \forall n$$