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Lectures Notes in Functinoal Analysis

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and since $A = A \cap \left(\bigcup_n \Omega_n\right) = \bigcup_n A \cap \Omega_n$

$$\mu(A) = \mu(\bigcup_n A \cap \Omega_n) \leq \sum_n \mu(A \cap \Omega_n) = 0.$$

So A is a null set and $\|u\|_\infty \leq \|\phi\|_{(L^1)^*}$. □

Claim:

$$\phi(h) = \int uh d\mu \quad \forall h \in L^1. \tag{$****$}$$

Indeed, choose $g = T_n h$ in $(***)$ and note that $\chi_n T_n h \to h$ in L^1.
Claim:

$$\|\phi\|_{(L^1)^*} = \|u\|_\infty.$$

Indeed, by $(****)$ one sees

$$|\phi(h)| \leq \|u\|_\infty \|h\|_1 \quad \forall h \in L^1$$

so $\|\phi\|_{(L^1)^*} \leq \|u\|_\infty$. □

Remark 8.10. *The space L^1 is never reflexive, except in the trivial case where Ω consists of a finite number of atoms. Then L^1 is finite-dimensional! Indeed, assume that L^1 is reflexive and consider two cases*

(i) $\forall \varepsilon > 0\ \exists A_\varepsilon \subset \Omega$ *measurable with* $0 < \mu(A_\varepsilon) < \varepsilon$.

(ii) $\exists \varepsilon > 0$ *such that* $\mu(A) \geq \varepsilon$ *for every measurable set* $A \subset \Omega$ *with* $\mu(A) > 0$.

In case (i) there exists a decreasing sequence A_n of measurable sets such that

$$0 < \mu(A_n) \to 0 \quad \text{as } n \to \infty.$$

(Choose first any sequence B_n such that

$$0 < \mu(B_n) < 2^{-n}$$

and set $A_n := \bigcup_{k=n}^\infty B_k$.)
Let $\chi_n := \mathbb{1}_{A_n}$ and set

$$u = \frac{\chi_n}{\|\chi_n\|_1}.$$

Since $\|u\|_1 = 1$ and since we assume that L^1 is reflexive, Theorem 7.28 applies and gives us a subsection (which we still denote by $(u_n)_n$) and $u \in L^1$ such that $u_n \rightharpoonup u$ weakly in L^1, i.e.,

$$\int u_n \phi d\mu \to \int u \phi d\mu \quad \forall \phi \in L^\infty.$$

Moreover, for fixed j and $n > j$ we have

$$\int_{A_j} u_n d\mu = \int u_n \chi_j d\mu = 1$$

so letting $n \to \infty$, we have

$$\int_{A_j} u d\mu = \int u\chi_j d\mu = \lim_{n\to\infty} \int u_n \chi_j d\mu = 1 \quad \forall j \in \mathbb{N}.$$

But, by dominated convergence, we have

$$\int u\chi_j d\mu \to 0 \quad as\ j \to \infty$$

which is a contradiction. So L^1 is not reflexive.
In case (ii) the space Ω is purely atomic and consists of a countable number of distinct atoms (a_n), unless there are only finitely many atoms. In this case, L^1 is isomorphic to $l^1(\mathbb{N})$ and we need only to show that l^1 is not reflexive. Consider the canonical basis

$$e_n = (0, \dots, 0, \underbrace{1}_{n-th\ slot}, 0, \dots).$$

Assuming that l^1 is reflexive, there exists a subsequence (e_{n_k}) and some $x \in l^1$ such that $e_{n_k} \rightharpoonup x$ in the weak topology $\sigma(l^1, l^\infty)$, i.e.

$$\underbrace{(\varphi, e_{n_k})}_{=\sum \varphi(j) e_{n_k}(j)} \to (\varphi, x) \quad \forall \varphi \in l^\infty.$$

Choosing $\varphi = \varphi_j = (0, 0, \dots, 0, \underbrace{1}_{j-th\ slot}, 1, \dots)$ we get

$$(\varphi_j, x) = \lim_{k\to\infty} \underbrace{(\varphi_j, e_{n_k})}_{=1\ \forall k \geq j} = 1$$

but

$$(\varphi_j, x) = \sum_{n\geq j} x(j) \to 0 \quad as\ j \to \infty,$$

since $x \in l^1$, a contradiction.

(C) Study of L^∞.
This is more complicated and we will not give a full answer. We already know $L^\infty = (L^1)^*$ by Theorem 8.9. Being a dual space, L^∞ has some nice properties, in particular

- The closed unit ball B_{L^∞} is compact in the weak$*$ topology $\sigma(L^\infty, L^1)$ by Theorem 7.2.

- If $\Omega \subset \mathbb{R}^d$ is measurable and $(f_n)_n$ is a bounded sequence in $L^\infty(\Omega)$, there exists a subsequence $(f_{n_k})_k$ and some $f \in L^\infty$ such that $f_{n_k} \rightharpoonup f$ in the weak$*$ topology $\sigma(L^\infty, L^1)$. This is a consequence of Corollary 7.42 which applies, since L^∞ is the dual space of the separable space L^1.

However, L^∞ is not reflexive, except in the case where Ω consists of a finite number of points, since otherwise $L^1(\Omega)$ were reflexive (since a Banach space E is reflexive if and only if E^* is reflexive), and we know by the previous discussion that L^1 is not reflexive (Remark 8.10)! Thus, the dual space $(L^\infty)^*$ contains L^1, since $L^\infty = (L^1)^*$, and $(L^\infty)^*$ is stricIty bigger than L^1. Thus there are continuous linear functionals ϕ on L^∞ which cannot be represented as

$$\phi(f) = \int u f d\mu \quad \forall f \in L^\infty \text{ and some } u \in L^1.$$

Example. *Let $\phi_0 : C_c(\mathbb{R}^d) \to \mathbb{R}$ (or $\mathbb{C}$) be defined by*

$$\phi_0(f) := f(0) \quad \forall f \in C_c(\mathbb{R}^d).$$

This is a continuous linear functional on $C_c(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and by Hahn-Banach, we may extend ϕ_0 to a continuous linear functional ϕ on $L^\infty(\mathbb{R}^d)$ and

$$\phi(f) = f(0) \quad \forall f \in C_c(\mathbb{R}^d).$$

BUT there is no $u \in L^1$ such that

$$\phi(f) = \int u f d\mu \quad \forall f \in L^\infty. \tag{$*$}$$

Assuming that such a function $u \in L^1$ exists, we get from $(*)$ that

$$\int u f dx = 0 \quad \forall f \in C_c(\mathbb{R}^d), f(0) = 0.$$

By some result from measure theorey, this implies that $u = 0$ a.e. on $\mathbb{R}^d \setminus \{0\}$, hence $u = 0$ a.e. on $\mathbb{R}^d$, but then

$$\phi(f) = \int u f d\mu = 0 \quad \forall f \in L^\infty,$$

a contradicion.

Remark. *In fact, the dual space of L^∞ is the space of (complex valued) Radon measures.*

Theorem 8.11. *$L^\infty(\mathbb{R}^d)$ is not separable. (In fact, $L^\infty(\Omega)$ is not separable, except if Ω consists of a finite number of atoms).*

Lemma 8.12. *Let E be a Banach space. Assume that there exists a family $(\mathcal{O}_i)_{i \in I} \subset E$ such that*

(a) $\forall i \in I, \mathcal{O}_i \neq \emptyset$ is open

(b) $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ *if* $i \neq j$

(c) I *is uncountable*

Then E *is not separable!*

Proof. Suppose that E is separable and let $(u_n)_{n\in\mathbb{N}}$ be a dense countable set in E. For each $i \in I$ the set $\mathcal{O}_i \cap (u_n)_{n\in\mathbb{N}} \neq \emptyset$ so we can choose $n(i)$ such that $u_{n(i)} \in \mathcal{O}_i$.
Note that the map $I \ni i \mapsto n(i) \in \mathbb{N}$ is injective, since, if $n(i) = n(j)$, then

$$u_{n(i)} = u_{n(j)} \in \mathcal{O}_i \cap \mathcal{O}_j$$

so by (b) we must have $i = j$!
Therefore, I is countable, a contradicion! □

Proof of Theorem 8.11. Let $I = \mathbb{R}^d$ and $\omega_i := B_1(i)$ (ball of radius one in $\mathbb{R}^d$ centered at $i \in \mathbb{R}^d$).
Note:

$$\omega_i \triangle \omega_j = (\omega_i \setminus \omega_j) \vee (\omega_j \setminus \omega_i) \neq 0 \quad \text{if } i \neq j.$$

Let

$$\mathcal{O}_i := \{f \in L^\infty(\mathbb{R}^d) | \; \|f - \mathbb{1}_{\omega_i}\|_\infty < \frac{1}{2}\}$$

and check that $(\mathcal{O}_i)_{i\in I}$ obeys the assumptions of Lemma 8.12 (for this note that $\|\mathbb{1}_{\omega_i} - \mathbb{1}_{\omega_j}\|_\infty = 1$ if $i \neq j$!) so by Lemma 8.12, L^∞ is not separable! □

	Reflexive	Separable	Dual space
$L^p, 1 < p < \infty$	YES	YES	$L^{p'}$
L^1	NO	YES	L^∞
L^∞	NO	NO	strictly bigger than L^1!

9 Hilbert spaces

9.1 Some elementary properties

Definition 9.1. *(a) Let* H *be a real vector space. A (real)* ***scalar product*** $< u, v >$ *on* H *is a bilinear form* $< \cdot, \cdot >: H \times H \to \mathbb{R}$ *that is linear in both variables such that* $\forall u, v \in H$

$$\begin{aligned} &< u, v > = < v, u > && \textit{(symmetry)} \\ &< u, u > \geq 0 && \textit{(positivity)} \\ &< u, u > = 0 \Rightarrow u = 0 \end{aligned}$$

(b) If H *is a complex vector space, a (complex) scalar product on* H *is a map* $< \cdot, \cdot >: H \times H \to \mathbb{C}$ *such that* $\forall u, w, x \in H, \alpha, \beta \in \mathbb{C}$*:*

$$\begin{aligned} < x, \alpha u + \beta w > &= \alpha < x, u > + \beta < x, w > \\ < u, w > &= \overline{< w, u >} \\ < u, u > &\geq 0 \quad \textit{and} < u, u >= 0 \Rightarrow u = 0 \end{aligned}$$

So $<\cdot,\cdot>$ is linear in the second argument and

$$\begin{aligned} <\alpha u+\beta w, x> &= \overline{<x, \alpha u+\beta w>} \\ &= \bar{\alpha}\overline{<x,u>}+\bar{\beta}\overline{<x,w>} \\ &= \bar{\alpha}<u,x>+\bar{\beta}<w,x> \end{aligned}$$

so it is "anti"-linear in the first component.

One always has the Cauchy-Schwarz inequality

$$|<u,v>| \leq <u,u>^{\frac{1}{2}}<v,v>^{\frac{1}{2}} .$$

Proof. W.l.o.g., $u, v \neq 0$.

$$\begin{aligned} 0 \leq &<tu-sv, tu-sv> = \bar{t}<u, tu-sv> - \bar{s}<v, tu-sv> \\ &= |t|^2<u,u> - \bar{t}s<u,v> - \bar{s}t\underbrace{<v,u>}_{=\overline{<u,v>}} + |s|^2<v,v> \\ &= |t|^2<u,u> + |s|^2<v,v> - 2Re(\bar{t}s<u,v>) \\ &= |t|^2<u,u> + |s|^2<v,v> - 2Re(\bar{t}se^{i\theta}|<u,v>|) \end{aligned}$$

where θ is such that $<u,v> = |<u,v>|e^{i\theta}$.
Choose $s = re^{-i\theta}, r, t > 0$ to get

$$0 \leq t^2<u,u> + r^2<v,v> - 2\underbrace{Re(tr|<u,v>|)}_{=tr|<u,v>|}$$

$$\Rightarrow |<u,v>| \leq \frac{1}{2}\Big(\frac{t}{r}<u,u> + \frac{r}{t}<v,v> - <tu-re^{-i\theta}v, tu-re^{-i\theta}v>\Big).$$

Now choose t, r such that $\lambda = \frac{t}{r} = \frac{<v,v>^{\frac{1}{2}}}{<u,u>^{\frac{1}{2}}}$

$$\Rightarrow |<u,v>| \leq <u,u>^{\frac{1}{2}}<v,v>^{\frac{1}{2}} - \frac{1}{2}\underbrace{<\ldots,\cdots>}_{\geq 0}$$

so we have the inequality, and if

$$|<u,v>| = <u,u>^{\frac{1}{2}}<v,v>^{\frac{1}{2}}$$

then we must have

$$<tu-re^{-i\theta}v, tu-re^{-i\theta}v> = 0$$

for some choice of $t, r > 0$. So $tu - re^{-i\theta}v = 0$, hence u and v are linearly dependant! □

Because of the Cauchy-Schwarz,

$$|u| := \sqrt{<u,u>} \quad \text{(the norm induced by } <\cdot,\cdot>)$$

is a norm (we write $|u|$ instead of $\|u\|$ if the norm comes from a scalar product). Indeed,

$$\begin{aligned} |u+v|^2 &= <u+v, u+v> = <u,u> + 2Re <u,v> + <v,v> \\ &\leq |u|^2 + 2| <u,v> | + |v|^2 \\ &\leq |u|^2 + 2|u||v| + |v|^2 \\ &= (|u| + |v|)^2 \end{aligned}$$

so

$$|u+v| \leq |u| + |v|.$$

Recall the parallelogram law

$$\begin{aligned} \left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2 &= \frac{1}{4}(<a+b, a+b> + <a-b, a-b>) \\ &= \frac{1}{4}(|a|^2 + <a,b> + <b,a> + |b|^2 \\ &\qquad + |a|^2 - <a,b> - <b,a> + |b|^2) \\ &= \frac{1}{2}(|a|^2 + |b|^2). \end{aligned}$$

Definition 9.2. *A Hilbert space is a (real or complex) vector space equipped with a scalar product $< \cdot, \cdot >$ such that H is complete w.r.t. the norm induced by $< \cdot, \cdot >$.*

Example.
- $L^2(\Omega)$ *with*

$$<u,v> := \int_\Omega \bar{u} v d\mu$$

is a Hilbert space.

- $l^2(\mathbb{N})$ *with*

$$<x,y> := \sum_{n \in \mathbb{N}} \bar{x_n} y_n$$

is a Hilbert space.

Proposition 9.3. *Any Hilbert space H is uniformly convex and thus reflexive.*

Proof. Let $\varepsilon > 0, u, v \in H, |u| \leq 1, |v| \leq 1$ and $|u-v| > \varepsilon$. Then, by the parallelogram law

$$\left|\frac{u+v}{2}\right|^2 \leq 1 - \left|\frac{u-v}{2}\right|^2 < 1 - \frac{\varepsilon^2}{4}$$

so

$$\left|\frac{u+v}{2}\right| \leq 1 - \delta$$

with $\delta = 1 - (1 - \frac{\varepsilon^2}{4})^{\frac{1}{2}} > 0$. $\square$

Theorem 9.4 (Projection theorem). *Let H be a Hilbert space and $K \subset H, K \neq \emptyset$, a closed convex set. Then for every $f \in H$ there exists a unique $u \in K$ such that*

$$|f-u| = \inf_{v\in K} |f-v| =: dist(f,K). \tag{1}$$

Moreover, u is characterized by the property

$$u \in K \quad and \quad Re < f-u, v-u > \leq 0 \ \forall v \in K. \tag{2}$$

Notation: The above element u is called **projection** of f onto K and is denoted by

$$u =: P_K f.$$

Proof. Existence: 1st proof: The function

$$K \ni v \mapsto \varphi(v) := |f-v|$$

is convex, continuous and

$$\lim_{v\in K, |v|\to\infty} \varphi(v) = \infty.$$

So by Corollary 7.33 we know that φ attains its minimum on K since H is reflexive.
2nd proof: Now a direct argument: Let $(v_n)_n \subset K$ be a minimizing sequence for (1), i.e., $v_n \in K$ and

$$d_n := |f-v_n| \to d := \inf_{v\in K} |f-v|.$$

Claim 1: $v := \lim_{n\to\infty} v_n$ exists and $v \in K$.
Indeed, apply the parallelogram identity to $a = f - v_n$ and $b = f - v_m$ to see

$$\left|f - \frac{v_n+v_m}{2}\right|^2 + \left|\frac{v_n-v_m}{2}\right|^2 = \frac{1}{2}(|f-v_n|^2 + |f-v_m|^2) = \frac{1}{2}(d_n^2+d_m^2).$$

Since K is convex, $\frac{v_n+v_m}{2} \in K$, so

$$\left|f - \frac{v_n+v_m}{2}\right|^2 \geq d^2$$

and hence

$$\left|\frac{v_n-v_m}{2}\right|^2 \leq \frac{1}{2}(d_n^2+d_m^2) - d^2 \to 0 \quad \text{as } n,m\to\infty$$

so

$$\lim_{n.m\to\infty} |v_n - v_m| = 0,$$

and $(v_n)_n$ is Cauchy! Thus $v = \lim_{n\to\infty} v_n$ exists and since K is closed, $v \in K$.
Equivalence of (1) and (2): Assume $u \in K$ satisfies (1) and let $w \in K$. Then

$$v := (1-t)u + tw \in K \quad \forall t \in [0,1]$$

so

$$|f-u| \leq |f-v| = |(f-u)-t(w-u)|$$

$$\Rightarrow |f-u|^2 \leq |f-u|^2 - 2tRe < f-u, w-u > +t^2|w-u|^2$$

so

$$\begin{aligned} 2Re < f-u, w-u > &\leq t|w-u|^2 \quad \forall t \in (0,1] \\ &\to 0 \quad \text{as } t \to 0 \end{aligned}$$

so (2) holds.
On the other hand, if (2) holds, then for $v \in K$,

$$\begin{aligned} |u-f|^2 - |v-f|^2 &= < u-f, u-f > - < v-f, v-f > \\ &= |u|^2 - 2Re < f, u > +|f|^2 - |v|^2 + 2Re < f, v > -|f|^2 \\ &= |u|^2 - |v|^2 + 2Re < f, v-u > \\ &= |u|^2 - |v|^2 + 2Re < f-u, v-u > +2Re < u, v-u > \\ &= -|u|^2 - |v|^2 + 2Re < u, v > +2Re < f-u, v-u > \\ &= -|u-v|^2 + \underbrace{2Re < f-u, v-u >}_{\leq 0 \text{ by } (2)} \leq 0, \end{aligned}$$

so (1) holds.
Uniqueness: Assume that $u_1, u_2 \in K$ satisfy (1). Then

$$Re < f-u_1, v-u_1 > \leq 0 \quad \forall v \in K \tag{3}$$

$$Re < f-u_2, v-u_2 > \leq 0 \quad \forall v \in K \tag{4}$$

Choose $v = u_2$ in (3) and $v = u_1$ in (4). Then

$$\begin{aligned} Re < f-u_1, u_2-u_1 > \leq 0, \\ Re < f-u_2, u_2-u_1 > \geq 0. \end{aligned}$$

$$\begin{aligned} \Rightarrow 0 &\geq Re < f-u_1, u_2-u_1 > -Re < f-u_2, u_2-u_1 > \\ &= Re < -u_1, u_2-u_1 > +Re < u_2, u_2-u_1 > \\ &= Re < u_2-u_1, u_2-u_1 > \\ &= |u_2-u_1|^2 \geq 0 \end{aligned}$$

so $|u_2 - u_1| = 0$, i.e., $u_2 = u_1$. □

Remark. *(1) It is not at all surprising to have a minimization problem related to a system of inequalities. Let $F : [0,1] \to \mathbb{R}$ be differentiable (with left and right derivatives at 1 and 0, resp.) and let $u \in [0,1]$ be a point at which F achieves its minimum. Then we have three cases:*

$$\begin{aligned} \textit{either } u \in (0,1) \quad &\textit{and} \quad F'(u) = 0 \\ \textit{or } u = 0 \quad &\textit{and} \quad F'(0) \geq 0 \\ \textit{or } u = 1 \quad &\textit{and} \quad F'(1) \leq 1 \end{aligned}$$