جامعة الانبار

كلية التربية للعلوم الصرفة

قسم الرياضيات / ماجستير

مقرر : التحليل الدالي

المحاضرة الاولى (11)

(المصدر)

Lectures Notes in Functinoal Analysis

WS 2012 – 2013

All three cases can be summarized as

$$u \in [0,1] \quad and \quad F'(u)(v-u) \geq 0 \quad \forall v \in [0,1].$$

(2) Let E be a uniformly convex Banach space, $K \subset E, K \neq \emptyset$, closed and convex. Then $\forall f \in E$ there exists a unique $u \in K$ such that

$$\|f-u\| = \inf_{v\in K} \|f-v\| =: dist(f,K).$$

Proposition 9.5. *Let $K \subset H, K \neq \emptyset$, closed and convex. Then P_K does not increase distance, i.e.,*

$$|P_K f_1 - P_K f_2| \leq |f_1 - f_2| \quad \forall f_1, f_2 \in H.$$

Proof. Let $u_j := P_K f_j$. Then as in the uniqueness proof of Theorem 9.4, we have by (2)

$$\begin{aligned} Re < f_1 - u_1, v - u_1 > &\leq 0 \quad \forall v \in K, \\ Re < f_2 - u_2, v - u_2 > &\leq 0 \quad \forall v \in K. \end{aligned}$$

Choose $v = u_2$ in the first inequality and $v = u_1$ in the second to see

$$\begin{aligned} Re < f_1 - u_1, u_2 - u_1 > &\leq 0, \\ Re < f_2 - u_2, u_2 - u_1 > &\geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &\geq Re < f_1 - u_1, u_2 - u_1 > + Re < f_2 - u_2, u_1 - u_2 > \\ &= Re < f_1 - u_1 - f_2 + u_2, u_2 - u_1 > \\ &= Re < f_1 - f_2, u_2 - u_1 > - |u_2 - u_1|^2. \end{aligned}$$

So

$$\begin{aligned} |u_2 - u_1|^2 &\leq Re < f_1 - f_2, u_2 - u_1 > \\ &\leq | < f_1 - f_2, u_2 - u_1 > | \\ &\underbrace{\leq}_{\text{CSI}} |f_1 - f_2||u_2 - u_1|. \end{aligned}$$

Thus

$$|u_2 - u_1| \leq |f_1 - f_2|.$$

□

Corollary 9.6. *Assume that $M \subset H$ is a linear subspace. Let $f \in H$. Then $u = P_M f$ is characterized by*

$$u \in M \quad and \quad < f - u, v > = 0 \quad \forall v \in M, \tag{6}$$

*i.e., $f - u$ is perpendicular to all $v \in M$. Moreover, P_M is a linear operator called the **orthogonal projection**.*

Proof. Step 1: By (2) we have

$$Re < f - u, v - u >= 0 \quad \forall v \in M.$$

Since M is a subspace, $tv \in M\ \forall t \in \mathbb{R}, v \in M$. Hence

$$\underbrace{Re < f - u, tv - u >}_{=tRe<f-u,v>-Re<f-u,v>} \leq 0 \quad \forall t \in \mathbb{R}$$

and thus for $t > 0$:

$$Re < f - u, v > \leq \frac{1}{t} Re < f - u, v > \to 0 \quad \text{as } t \to \infty$$

so

$$Re < f - u, v > \leq 0$$

and for $t < 0$:

$$Re < f - u, v > \geq \frac{1}{t} Re < f - u, v > \to 0 \quad \text{as } t \to \infty$$

so

$$Re < f - u, v > \leq 0 \quad \text{and} \quad Re < f - u, v > \geq 0,$$

i.e.,

$$Re < f - u, v >= 0 \quad \forall v \in M.$$

Replace v by $-iv$. Then

$$0 = Re < f - u, -iv >= Re(-i < f - u, v >) = Im < f - u, v >$$

so (6) holds.
Step 2:

$$|P_M f| \leq |f| \quad \forall f \in H.$$

Indeed, since M is linear, $0 \in M$ and $P_M 0 = 0$, so by Proposition 9.5

$$|P_M f| = |P_M f - P_M 0| \leq |f - 0| = |f|.$$

Step 3: If u satisfies (6), then $u = P_M f$.
Indeed, if

$$< f - u, v >= 0 \quad \forall v \in M,$$

then, since $u \in M$, and M is linear, $v - u \in M$, so

$$< f - u, v - u >= 0.$$

Hence (2) holds which characterizes $u = P_M f$!
Step 4: $P_M : H \to M$ is linear.
Indeed, if $f_1, f_2 \in H, u_j = P_M f_j, \alpha_1, \alpha_2 \in \mathbb{F}$, then

$$< f_1 - u_1, v >= 0 \quad \forall v \in M,$$
$$< f_2 - u_2, v >= 0 \quad \forall v \in M.$$

Thus

$$\begin{aligned}0 &=< \alpha_1 f_1 - \alpha_1 u_1, v > + < \alpha_2 f_2 - \alpha_2 u_2, v > \\ &=< \alpha_1 f_1 + \alpha_2 f_2 - (\alpha_1 u_1 + \alpha_2 u_2), v >,\end{aligned}$$

i.e.,

$$\alpha_1 u_1 + \alpha_2 u_2 = P_M(\alpha_1 f_1 + \alpha_2 f_2).$$

□

9.2 The dual space of a Hilbert space

There are plenty of continuous linear functionals on a Hilbert space H. Simply pick $f \in H$ and consider

$$u \mapsto < f, u > .$$

The remarkable fact is that all continuous linear functionals on H are of this form!

Theorem 9.7 (Riesz-Fréchet representation theorem). *Given any $\varphi \in H^*$ there exists a unique $f = f_\varphi \in H$ such that*

$$\varphi(u) =< f, u > \quad \forall u \in H.$$

Moreover,

$$|f| = \|\varphi\|_{H^*}.$$

Proof. 1st: Consider the map $T : H \to H^*$,

$$Tf :=< f, \cdot > \in H^*,$$

i.e.,

$$Tf(u) :=< f, u > .$$

It is clear that $\|Tf\|_{H^*} = |f|$ (why?), so T is an isometry from H onto $T(H)$, i.e., $T(H)$ is a closed subspace of H^*. Assume $h \in (H^*)^*$ which vanishes on $T(H)$. Since H is reflexive, $h \in H$ and

$$Tf(h) =< f, h >= 0 \quad \forall f \in H.$$

Take $f = h$. Then

$$|h|^2 =< h, h >= 0 \quad \Rightarrow \quad h = 0,$$

i.e., $T(H)$ is dense in H^* and thus $T(H) = H$.
2nd: Given $\varphi \in H^*$, let

$$M := \varphi^{-1}(\{0\}) \subset H$$

and note that M is closed since φ is continuous.
Assume $M \neq H$ (otherwise $\varphi \equiv 0$ and we take $f = 0$). Pick any $g_0 \in H$ such that $\varphi(g_0) \neq 0$ and set $g_1 := P_M g_0 \in M$. Note

$$\varphi(g_0 - g_1) = \varphi(g_0) - \underbrace{\varphi(g_1)}_{=0} = \varphi(g_0) \neq 0$$

so

$$g_0 - g_1 \neq 0.$$

Put

$$g := \frac{g_0 - g_1}{|g_0 - g_1|}.$$

Then $|g| = 1$,

$$\varphi(g) = \frac{\varphi(g_0)}{|g_0 - g_1|} \neq 0$$

and

$$< g, v >= \frac{1}{|g_0 - g_1|} < g_0 - g_1, v >= \frac{1}{|g_0 - g_1|} < g_0 - P_M g_0, v >= 0$$

by Corollary 9.6.

Given $u \in H$ let

$$v = u - \lambda g$$

and choose λ such that $v \in M$, i.e.,

$$\lambda = \frac{\varphi(u)}{\varphi(g)}.$$

But then

$$\begin{aligned} 0 &=< g, v >=< g, u - \lambda g > \\ &=< g, v > -\lambda \underbrace{< g, g >}_{=1} \\ &=< g, v > - \frac{\varphi(u)}{\varphi(g)}. \end{aligned}$$

Thus

$$\varphi(u) = \varphi(g) < g, u >=< \overline{\varphi(g)} g, u >$$

so $f := \overline{\varphi(g)}$ works. □

9.3 The Theorems of Stampacchia and Lax-Milgram

In the following, let H be a real Hilbert space.

Definition. *A bilinear form $a : H \times H \to \mathbb{R}$ is said to be*

(i) continuous, if there exists $C > 0$ such that

$$|a(u,v)| \le C|u||v| \quad \forall u, v \in H;$$

(ii) coercive, if there exists $\alpha > 0$ such that

$$a(v,v) \ge \alpha |v|^2 \quad \forall v \in H.$$

Theorem 9.8 (Stampacchia)**.** *Assume that a is a continuous coercive bilinear form on a real Hilbert space H. Let $K \subset H, K \neq \emptyset$ closed and convex. Then given $\varphi \in H^*$ there exists a unique $u \in K$ such that*

$$a(u, v-u) \ge \varphi(v-u) \quad \forall v \in K. \tag{1}$$

Moreover, if a is symmetric, then u is characterized by

$$u \in K \quad \text{and} \quad \frac{1}{2}a(u,u) - \varphi(u) = \inf_{v \in K} \Big(\frac{1}{2}a(v,v) - \varphi(v)\Big). \tag{2}$$

We need

Theorem 9.9 (Banach fixed point theorem)**.** *Let $X \neq \emptyset$ be a complete metric space and $S : X \to X$ be a strict contraction, i.e.,*

$$d(S(x_1), S(x_2)) \le k d(x_1, x_2) \quad \forall x_1, x_2 \in X \text{ with } k < 1.$$

Then S has a unique fixed point u, i.e., $u = S(u)$.

Proof of Theorem 9.8. By Riesz representation theorem there exists $f \in H$ such that

$$\varphi(v) = < f, v > \quad \forall v \in H.$$

Note that also the maps $v \mapsto a(u,v) \in H^*$, so again there exists a unique element in H, denoted by Au such that

$$a(u,v) = < Au, v > \quad \forall v \in H.$$

<u>Note:</u> A is a linear operator from H to H and

$$|Au| \le C|u| \quad \forall u \in H,$$
$$< Au, u > \ge \alpha |u|^2 \quad \forall u \in H.$$

So problem (1) says we should find $u \in K$ such that

$$< Au, v-u > \ge < f, v-u > \quad \forall v \in K. \tag{3}$$

Let $\rho > 0$ and note that (3) is equivalent to

$$< \rho f - \rho Au + u - u, v - u > \le 0 \quad \forall v \in K, \tag{4}$$

i.e.,

$$u = P_K(\rho f - \rho Au + u).$$

For $v \in K$ set

$$S(v) := P_K(\rho f - \rho Av + v).$$

Claim: Choosing $\rho > 0$ cleverly, S is a strict contraction, so it has a unique fixed point!
Indeed,

$$\begin{aligned}|Sv_1 - Sv_2| &= |P_K(\rho f - \rho Av_1 + v_1) - P_K(\rho f - \rho Av_2 + v_2)| \\ &\leq |\rho f - \rho Av_1 + v_1 - \rho f + \rho Av_2 - v_2| \\ &= |(v_1 - v_2) + \rho(Av_1 - Av_2)|\end{aligned}$$

$$\begin{aligned}\Rightarrow |Sv_1 - Sv_2|^2 &= |v_1 - v_2|^2 - 2\rho \underbrace{< Av_1 - Av_2, v_1 - v_2 >}_{\geq \alpha |v_1 - v_2|^2} + \rho^2 |Av_1 - Av_2|^2 \\ &\leq |v_1 - v_2|^2 (1 - 2\rho\alpha + \rho^2 C^2).\end{aligned}$$

Choose ρ so that

$$K^2 = 1 - 2\rho\alpha + \rho^2 C^2 < 1,$$

i.e., $0 < \rho < \frac{2\alpha}{C^2}$. Then S has a unique fixed point.

Assume now that a is symmetric. Then $a(u, v)$ defines a new scalar product on H with norm $\sqrt{a(u, u)}$ which is equivalent to the old norm $|u|$. Thus H is a Hilber space for this new scalar product. By Riesz-Fréchet for $a(u, v)$, it follows that given $\varphi \in H^*$ there exists a unique $g \in H$ such that

$$\varphi(u) = a(g, u) \quad \forall u \in H.$$

Note that problem (1) amounts to finding some $u \in K$ such that

$$a(g - u, v - u) \leq 0 \quad \forall v \in K \tag{5}$$

but the solution to (5) is the projection onto K of g for the new scalar product a! By Theorem 9.4 $u \in K$ is the unique element which achieves

$$\inf \sqrt{a(g - v, g - v)},$$

i.e., one minimizes on K the function

$$\begin{aligned}v \mapsto a(g - v, g - v) &= a(v, v) - 2a(g, v) + a(g, g) \\ &= a(v, v) - 2\varphi(u) + a(g, g)\end{aligned}$$

or equivalently, the function

$$v \mapsto \frac{1}{2} a(v, v) - \varphi(u).$$

□

Corollary 9.10 (Lax-Milgram)**.** *Assume that $a(u,v)$ is a continuous coercive bilinear form on H. Then given any $\varphi \in H^*$ there exists a unique $u \in H$ such that*

$$a(u,v) = \varphi(u) \quad \forall v \in H. \tag{6}$$

Moreover, if a is symmetric, then u is characterized by

$$u \in H \quad \textit{and} \quad \frac{1}{2}a(u,u) - \varphi(u) = \inf_{\frac{1}{2}v \in H} (a(v,v) - \varphi(v)). \tag{7}$$

Proof. Apply Theorem 9.8 with $K = H$ and use linearity of H as in the proof of Corollary 9.6. □