Chapter 1

Mathematical Preliminaries and Error Analysis

1.1 Introduction

This book examines problems that can be solved by methods of approximation, techniques we call *numerical methods*. We begin by considering some of the mathematical and computational topics that arise when approximating a solution to a problem.

Nearly all the problems whose solutions can be approximated involve continuous functions, so calculus is the principal tool to use for deriving numerical methods and verifying that they solve the problems. The calculus definitions and results included in the next section provide a handy reference when these concepts are needed later in the book.

There are two things to consider when applying a numerical technique to solve a problem. The first and most obvious is to obtain the approximation. The equally important second objective is to determine a safety factor for the approximation: some assurance, or at least a sense, of the accuracy of the approximation. Sections 1.3 and 1.4 deal with a standard difficulty that occurs when applying techniques to approximate the solution to a problem: Where and why is computational error produced and how can it be controlled?

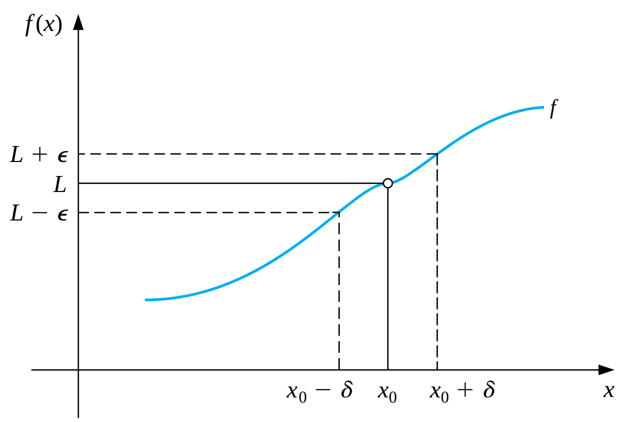
The final section in this chapter describes various types and sources of mathematical software for implementing numerical methods.

1.2 Review of Calculus

The limit of a function at a specific number tells, in essence, what the function values approach as the numbers in the domain approach the specific number. This is a difficult concept to state precisely. The limit concept is basic to calculus, and the major developments of calculus were discovered in the latter part of the seventeenth century, primarily by Isaac Newton and Gottfried Leibnitz. However, it was not until 200 years later that Augustus Cauchy, based on work of Karl Weierstrass, first expressed the limit concept in the form we now use.

We say that a function f defined on a set X of real numbers has the **limit** L at x_0, written $\lim_{x \to x_0} f(x) = L$, if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - x_0| < \delta$. This definition ensures that values of the function will be close to L whenever x is sufficiently close to x_0. (See Figure 1.1.)

Figure 1.1



A function is said to be continuous at a number in its domain when the limit at the number agrees with the value of the function at the number. So, a function f is **continuous** at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$, and f is **continuous on the set** X if it is continuous at each number in X. We use $C(X)$ to denote the set of all functions that are continuous on X. When X is an interval of the real line, the parentheses in this notation are omitted. For example, the set of all functions that are continuous on the closed interval $[a, b]$ is denoted $C[a, b]$.

The limit of a sequence of real or complex numbers is defined in a similar manner. An infinite sequence $\{x_n\}_{n=1}^{\infty}$ **converges** to a number x if, given any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$ whenever $n > N(\varepsilon)$. The notation $\lim_{n \to \infty} x_n = x$, or $x_n \to x$ as $n \to \infty$, means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

[Continuity and Sequence Convergence] If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following are equivalent:

a. f is continuous at x_0;

b. If $\{x_n\}_{n=1}^{\infty}$ is any sequence in X converging to x_0, then

$$\lim_{n \to \infty} f(x_n) = f(x_0).$$

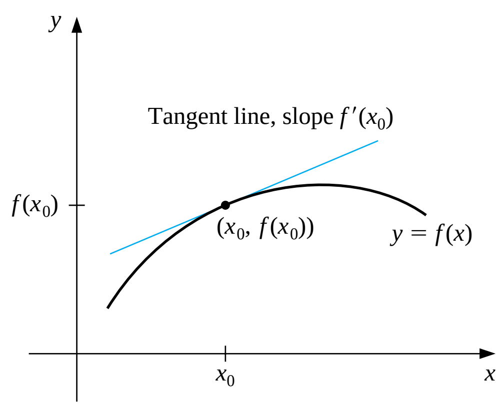
All the functions we will consider when discussing numerical methods will be continuous since this is a minimal requirement for predictable behavior. Functions that are not continuous can skip over points of interest, which can cause difficulties when we attempt to approximate a solution to a problem. More sophisticated assumptions about a function generally lead to better approximation results.For example, a function with a smooth graph would normally behave more predictably than one with numerous jagged features. Smoothness relies on the concept of the derivative.

If f is a function defined in an open interval containing x_0, then f is **differentiable** at x_0 when

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the **derivative** of f at x_0. The derivative of f at x_0 is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$, as shown in Figure 1.2.

Figure 1.2



A function that has a derivative at each number in a set X is **differentiable** on X. Differentiability is a stronger condition on a function than continuity in the following sense.

> [Differentiability Implies Continuity] If the function f is differentiable at x_0, then f is continuous at x_0.

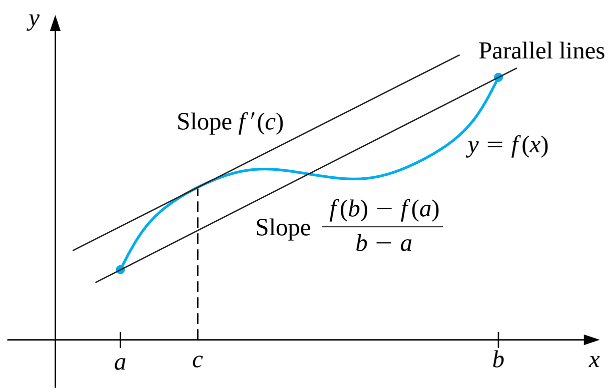
The set of all functions that have n continuous derivatives on X is denoted $C^n(X)$, and the set of functions that have derivatives of all orders on X is denoted $C^\infty(X)$. Polynomial, rational, trigonometric, exponential, and logarithmic functions are in $C^\infty(X)$, where X consists of all numbers at which the function is defined.

The next results are of fundamental importance in deriving methods for error estimation. The proofs of most of these can be found in any standard calculus text.

[Mean Value Theorem] If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists such that (see Figure 1.3)

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Figure 1.3



The following result is frequently used to determine bounds for error formulas.

[Extreme Value Theorem] If $f \in C[a,b]$, then c_1 and c_2 in $[a,b]$ exist with $f(c_1) \le f(x) \le f(c_2)$ for all x in $[a,b]$. If, in addition, f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at endpoints of $[a,b]$ or where f' is zero.

As mentioned in the preface, we will use the computer algebra system Maple whenever appropriate. We have found this package to be particularly useful for symbolic differentiation and plotting graphs. Both techniques are illustrated in Example 1.

EXAMPLE 1 Use Maple to find $\max_{a\leq x\leq b}|f(x)|$ for

$$f(x) = 5\cos 2x - 2x\sin 2x,$$

on the intervals $[1, 2]$ and $[0.5, 1]$.

We will first illustrate the graphing capabilities of Maple. To access the graphing package, enter the command

```
>with(plots);
```

A list of the commands within the package are then displayed. We define f by entering

```
>f:= 5*cos(2*x)-2*x*sin(2*x);
```

The response from Maple is

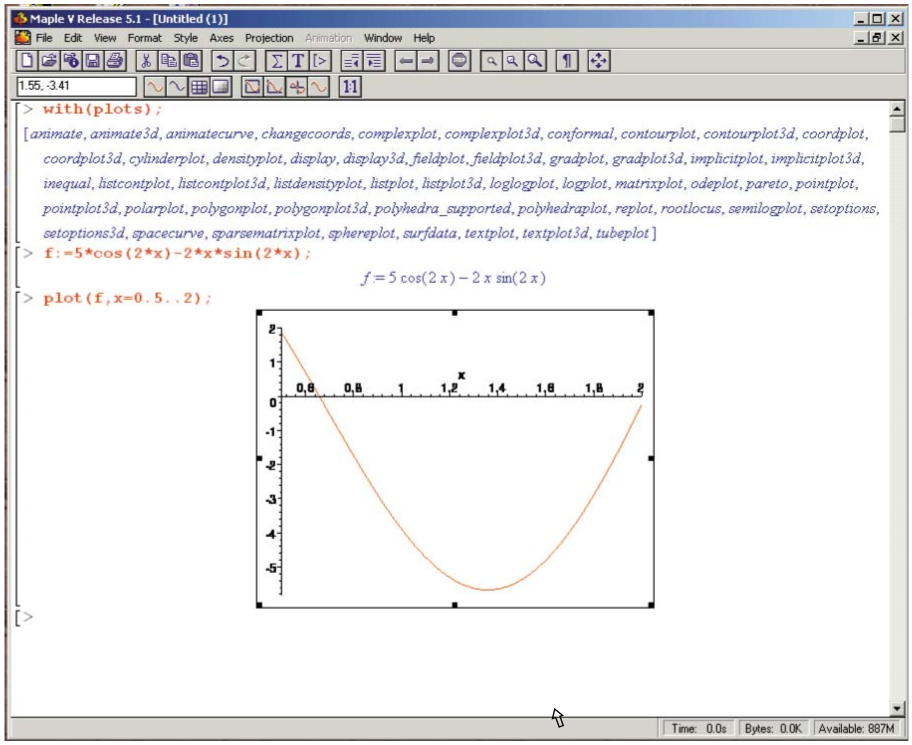
$$f := 5\cos(2x) - 2x\sin(2x)$$

To graph f on the interval $[0.5, 2]$, use the command

```
>plot(f,x=0.5..2);
```

We can determine the coordinates of a point of the graph by moving the mouse cursor to the point and clicking the left mouse button. The coordinates of the point to which the cursor is pointing appear in the white box at the upper left corner of the Maple screen, as shown in Figure 1.4. This technique is useful for estimating the axis intercepts and extrema of functions.

Figure 1.4



We complete the example using the Extreme Value Theorem. First, consider the interval $[1, 2]$. To obtain the first derivative, $g = f'$, we enter

```
>g:=diff(f,x);
```

Maple returns

$$g := -12\sin(2x) - 4x\cos(2x)$$

We can then solve $g(x) = 0$ for $1 \leq x \leq 2$ with the statement

```
>fsolve(g,x,1..2);
```

obtaining 1.358229874, and compute $f(1.358229874) = -5.675301338$ using

```
>evalf(subs(x=1.358229874,f));
```

This implies that we have a minimum of approximately $f(1.358229874) = -5.675301338$.

What we will frequently need is the maximum magnitude that a function can attain on an interval. This maximum magnitude will occur at a critical point or at an endpoint. Since $f(1) = -3.899329037$ and $f(2) = -0.241008124$, the maximum magnitude occurs at the critical point and

$$\max_{1\le x\le 2} |f(x)| = \max_{1\le x\le 2} |5\cos 2x - 2x\sin 2x| \approx |f(1.358229874)| = 5.675301338.$$

If we try to solve $g(x) = 0$ for $0.5 \le x \le 1$, we find that when we enter

```
>fsolve(g,x,0.5..1);
```

Maple responds with

$$\text{fsolve}(-12\sin(2x) - 4x\cos(2x), x, .5..1)$$

This indicates that Maple could not find a solution in $[0.5, 1]$, for the very good reason that there is no solution in this interval. As a consequence, the maximum occurs at an endpoint on the interval $[0.5, 1]$. Since $f(0.5) = 1.860040545$ and $f(1) = -3.899329037$, we have

$$\max_{0.5\le x\le 1} |f(x)| = \max_{0.5\le x\le 1} |5\cos 2x - 2x\sin 2x| = |f(1)| = 3.899329037.$$

□

The integral is the other basic concept of calculus that is used extensively. The **Riemann integral** of the function f on the interval $[a, b]$ is the following limit, provided it exists.

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(z_i)\,\Delta x_i,$$

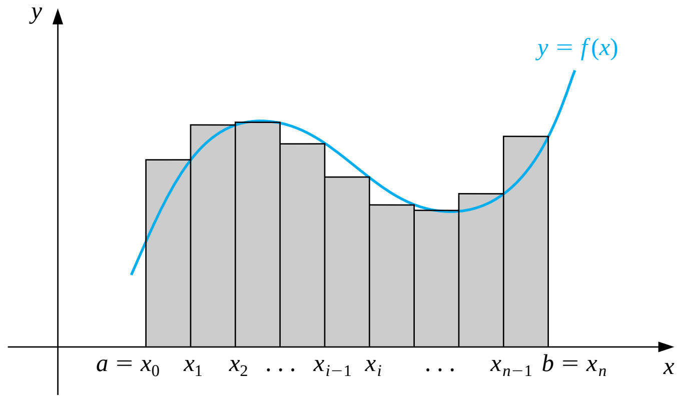
where the numbers $x_0, x_1, \ldots, x_n$ satisfy $a = x_0 < x_1 < \cdots < x_n = b$ and where $\Delta x_i = x_i - x_{i-1}$, for each $i = 1, 2, \ldots, n$, and z_i is arbitrarily chosen in the interval $[x_{i-1}, x_i]$.

A function f that is continuous on an interval $[a, b]$ is also Riemann integrable on $[a, b]$. This permits us to choose, for computational convenience, the points x_i to be equally spaced in $[a, b]$ and for each $i = 1, 2, \ldots, n$, to choose $z_i = x_i$. In this case

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i),$$

where the numbers shown in Figure 1.5 as x_i are $x_i = a + (i(b-a)/n)$.

Figure 1.5



Two more basic results are needed in our study of numerical methods. The first is a generalization of the usual Mean Value Theorem for Integrals.

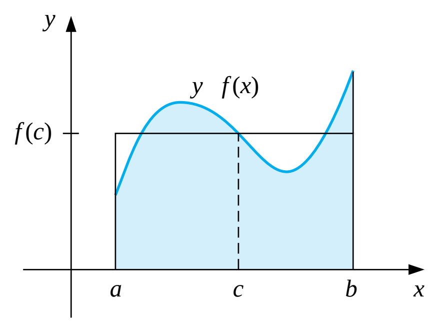
[Mean Value Theorem for Integrals] If $f \in C[a,b]$, g is integrable on $[a,b]$ and $g(x)$ does not change sign on $[a,b]$, then there exists a number c in (a,b) with

$$\int_a^b f(x)g(x)\ dx = f(c)\int_a^b g(x)\ dx.$$

When $g(x) \equiv 1$, this result reduces to the usual Mean Value Theorem for Integrals. It gives the **average value** of the function f over the interval $[a,b]$ as

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

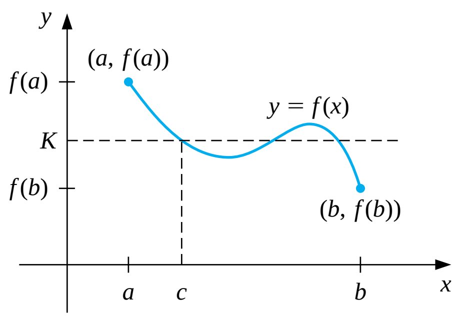
(See Figure 1.6.)
Figure 1.6



The next theorem presented is the Intermediate Value Theorem. Although its statement is not difficult, the proof is beyond the scope of the usual calculus course.

[Intermediate Value Theorem] If $f \in C[a,b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a,b) for which $f(c) = K$. (Figure 1.7 shows one of the three possibilities for this function and interval.)

Figure 1.7



EXAMPLE 2 To show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval $[0,1]$, consider $f(x) = x^5 - 2x^3 + 3x^2 - 1$. We have

$$f(0) = -1 < 0 \quad \text{and} \quad 0 < 1 = f(1),$$

and f is continuous. Hence, the Intermediate Value Theorem implies a number x exists, with $0 < x < 1$, for which $x^5 - 2x^3 + 3x^2 - 1 = 0$. □

As seen in Example 2, the Intermediate Value Theorem is used to help determine when solutions to certain problems exist. It does not, however, give an efficient means for finding these solutions. This topic is considered in Chapter 2.

The final theorem in this review from calculus describes the development of the Taylor polynomials. The importance of the Taylor polynomials to the study of numerical analysis cannot be overemphasized, and the following result is used repeatedly.

[Taylor's Theorem] Suppose $f \in C^n[a,b]$ and $f^{(n+1)}$ exists on $[a,b]$. Let x_0 be a number in $[a,b]$. For every x in $[a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \end{aligned}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}.$$

Here $P_n(x)$ is called the n**th Taylor polynomial** for f about x_0, and $R_n(x)$ is called the **truncation error** (or *remainder term*) associated with $P_n(x)$. Since the number $\xi(x)$ in the truncation error $R_n(x)$ depends on the value of x at which the polynomial $P_n(x)$ is being evaluated, it is actually a function of the variable x. However, we should not expect to be able to explicitly determine the function $\xi(x)$. Taylor's Theorem simply ensures that such a function exists, and that its value lies between x and x_0. In fact, one of the common problems in numerical methods is to try to determine a realistic bound for the value of $f^{(n+1)}(\xi(x))$ for values of x within some specified interval.

The infinite series obtained by taking the limit of $P_n(x)$ as $n \to \infty$ is called the *Taylor series* for f about x_0. In the case $x_0 = 0$, the Taylor polynomial is often called a **Maclaurin polynomial**, and the Taylor series is called a *Maclaurin series*.

The term *truncation error* in the Taylor polynomial refers to the error involved in using a truncated (that is, finite) summation to approximate the sum of an infinite series.

EXAMPLE 3 Determine (a) the second and (b) the third Taylor polynomials for $f(x) = \cos x$ about $x_0 = 0$, and use these polynomials to approximate $\cos(0.01)$. (c) Use the third Taylor polynomial and its remainder term to approximate $\int_0^{0.1} \cos x\, dx$.

Since $f \in C^\infty(\mathbb{R})$, Taylor's Theorem can be applied for any $n \geq 0$. Also,

$$f'(x) = -\sin x,\ f''(x) = -\cos x,\ f'''(x) = \sin x, \quad \text{and} \quad f^{(4)}(x) = \cos x,$$

so

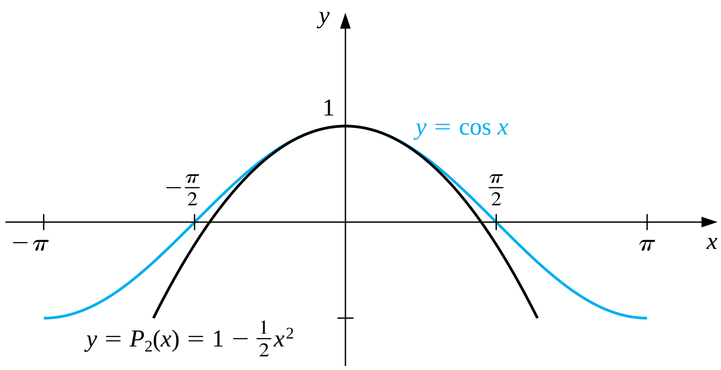
$$f(0) = 1,\ f'(0) = 0,\ f''(0) = -1, \quad \text{and} \quad f'''(0) = 0.$$

a.For $n = 2$ and $x_0 = 0$, we have

$$\begin{aligned} \cos x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3 \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin \xi(x), \end{aligned}$$

where $\xi(x)$ is some (unknown) number between 0 and x. (See Figure 1.8.)

Figure 1.8



When $x = 0.01$, this becomes

$$\cos 0.01 = 1 - \frac{1}{2}(0.01)^2 + \frac{1}{6}(0.01)^3 \sin \xi(0.01) = 0.99995 + \frac{10^{-6}}{6} \sin \xi(0.01).$$

The approximation to $\cos 0.01$ given by the Taylor polynomial is therefore 0.99995. The truncation error, or remainder term, associated with this approximation is

$$\frac{10^{-6}}{6} \sin \xi(0.01) = 0.1\overline{6} \times 10^{-6} \sin \xi(0.01),$$

where the bar over the 6 in $0.1\overline{6}$ is used to indicate that this digit repeats indefinitely. Although we have no way of determining $\sin \xi(0.01)$, we know that all values of the sine lie in the interval $[-1, 1]$, so the error occurring if we use the approximation 0.99995 for the value of $\cos 0.01$ is bounded by

$$|\cos(0.01) - 0.99995| = 0.1\overline{6} \times 10^{-6} \sin \xi(0.01) \leq 0.1\overline{6} \times 10^{-6}.$$

Hence the approximation 0.99995 matches at least the first five digits of $\cos 0.01$. Using standard tables we find that $\cos 0.01 = 0.99995000042$, so the approximation actually gives agreement through the first nine digits.

The error bound is much larger than the actual error. This is due in part to the poor bound we used for $|\sin \xi(x)|$.

It can be shown that or all values of x, we have $|\sin x| \le |x|$. Since $0 < \xi(x) < 0.01$, we could have used the fact that $|\sin\xi(x)| \le 0.01$ in the error formula, producing the bound $0.1\overline{6} \times 10^{-8}$.

b. Since $f'''(0) = 0$, the third Taylor polynomial and remainder term about $x_0 = 0$ are

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \cos\tilde{\xi}(x),$$

where $\tilde{\xi}(x)$ is some number between 0 and x, and likely distinct from the value of $\xi(x)$ that is associated with the remainder term of the second Taylor polynomial.

Notice that the second and third Taylor polynomials are the same, so the approximation to $\cos 0.01$ is still 0.99995. However, we now have a much better accuracy assurance. Since $|\cos\tilde{\xi}(x)| \le 1$ for all x, when $x = 0.01$ we have

$$\left|\frac{1}{24}x^4 \cos\tilde{\xi}(x)\right| \le \frac{1}{24}(0.01)^4(1) \approx 4.2 \times 10^{-10}.$$

The first two parts of the example illustrate the two objectives of numerical analysis:

(i) Find an approximation to the solution of a given problem.

(ii) Determine a bound for the accuracy of the approximation.

The Taylor polynomials in both parts provide the same answer to (i), but the third Taylor polynomial gave a much better answer to (ii) than the second Taylor polynomial.

c. Using the third Taylor polynomial gives

$$\begin{aligned}\int_0^{0.1} \cos x \, dx &= \int_0^{0.1} \left(1 - \frac{1}{2}x^2\right) dx + \frac{1}{24}\int_0^{0.1} x^4 \cos\tilde{\xi}(x) \, dx \\ &= \left[x - \frac{1}{6}x^3\right]_0^{0.1} + \frac{1}{24}\int_0^{0.1} x^4 \cos\tilde{\xi}(x) \, dx \\ &= 0.1 - \frac{1}{6}(0.1)^3 + \frac{1}{24}\int_0^{0.1} x^4 \cos\tilde{\xi}(x) \, dx.\end{aligned}$$

Therefore,

$$\int_0^{0.1} \cos x \, dx \approx 0.1 - \frac{1}{6}(0.1)^3 = 0.0998\overline{3}.$$

A bound for the error in this approximation is determined from the integral of the Taylor remainder term and the fact that $|\cos\tilde{\xi}(x)| \le 1$ for all x:

$$\frac{1}{24}\left|\int_0^{0.1} x^4 \cos\tilde{\xi}(x) \, dx\right| \le \frac{1}{24}\int_0^{0.1} x^4 |\cos\tilde{\xi}(x)| \, dx \le \frac{1}{24}\int_0^{0.1} x^4 \, dx = 8.\overline{3} \times 10^{-8}.$$

The true value of this integral can be easily determined as

$$\int_0^{0.1} \cos x \, dx = \sin x \Big]_0^{0.1} = \sin 0.1.$$

The true value of $\sin 0.1$ to nine decimal places is 0.099833417, so the approximation derived from the Taylor polynomial is in error by

$$|0.099833417 - 0.09983\overline{3}| \approx 8.4 \times 10^{-8},$$

which is essentially the same as the error bound derived from the Taylor polynomial. □

We can use a computer algebra system to simplify the calculations in Example 3. In the system Maple, we define f by

```
>f:=cos(x);
```

Maple allows us to place multiple statements on a line, and to use a colon to suppress Maple responses. For example, we obtain the third Taylor polynomial with

```
>s3:=taylor(f,x=0,4): p3:=convert(s3, polynom);
```

The statement `s3:=taylor(f,x=0,4)` determines the Taylor polynomial about $x_0 = 0$ with four terms (degree 3) and its remainder. The statement `p3:=convert(s3, polynom)` converts the series `s3` to the polynomial `p3` by dropping the remainder. To obtain 11 decimal digits of display, we enter

```
>Digits:=11;
```

and evaluate $f(0.01)$, $P_3(0.01)$, and $|f(0.01) - P_3(0.01)|$ with

```
>y1:=evalf(subs(x=0.01,f));
>y2:=evalf(subs(x=0.01,p3));
>err:=abs(y1-y2);
```

This produces $y_1 = f(0.01) = 0.99995000042$, $y_2 = P_3(0.01) = 0.99995000000$, and $|f(0.01) - P_3(0.01)| = .42 \times 10^{-9}$.

To obtain a graph similar to Figure 1.8, enter

```
>plot({f,p3},x=-Pi..Pi);
```

The commands for the integrals are

```
>q1:=int(f,x=0..0.1);
>q2:=int(p3,x=0..0.1);
>err:=abs(q1-q2);
```

which give the values

$$q_1 = \int_0^{0.1} f(x)\,dx = 0.099833416647 \quad \text{and} \quad q_2 = \int_0^{0.1} P_3(x)\,dx = 0.099833333333,$$

with error $0.83314 \times 10^{-7} = 8.3314 \times 10^{-8}$.

Parts (a) and (b) of Example 3 show how two techniques can produce the same approximation but have differing accuracy assurances. Remember that determining approximations is only part of our objective. The equally important other part is to determine at least a bound for the accuracy of the approximation.

EXERCISE SET 1.2

1. Show that the following equations have at least one solution in the given intervals.

   (a) $x\cos x - 2x^2 + 3x - 1 = 0, \quad [0.2, 0.3]$ and $[1.2, 1.3]$

   (b) $(x-2)^2 - \ln x = 0, \quad [1, 2]$ and $[e, 4]$

   (c) $2x\cos(2x) - (x-2)^2 = 0, \quad [2, 3]$ and $[3, 4]$

   (d) $x - (\ln x)^x = 0, \quad [4, 5]$

2. Find intervals containing solutions to the following equations.

   (a) $x - 3^{-x} = 0$

   (b) $4x^2 - e^x = 0$

   (c) $x^3 - 2x^2 - 4x + 3 = 0$

   (d) $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$

3. Show that the first derivatives of the following functions are zero at least once in the given intervals.

   (a) $f(x) = 1 - e^x + (e-1)\sin((\pi/2)x), \quad [0, 1]$

   (b) $f(x) = (x-1)\tan x + x\sin \pi x, \quad [0, 1]$

   (c) $f(x) = x\sin \pi x - (x-2)\ln x, \quad [1, 2]$

   (d) $f(x) = (x-2)\sin x \ln(x+2), \quad [-1, 3]$

4. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

   (a) $f(x) = (2 - e^x + 2x)/3, \quad [0, 1]$

   (b) $f(x) = (4x-3)/(x^2 - 2x), \quad [0.5, 1]$

   (c) $f(x) = 2x\cos(2x) - (x-2)^2, \quad [2, 4]$

   (d) $f(x) = 1 + e^{-\cos(x-1)}, \quad [1, 2]$

5. Let $f(x) = x^3$.

   (a) Find the second Taylor polynomial $P_2(x)$ about $x_0 = 0$.

   (b) Find $R_2(0.5)$ and the actual error when using $P_2(0.5)$ to approximate $f(0.5)$.

   (c) Repeat part (a) with $x_0 = 1$.

   (d) Repeat part (b) for the polynomial found in part (c).

6. Let $f(x) = \sqrt{x+1}$.

   (a) Find the third Taylor polynomial $P_3(x)$ about $x_0 = 0$.

   (b) Use $P_3(x)$ to approximate $\sqrt{0.5}$, $\sqrt{0.75}$, $\sqrt{1.25}$, and $\sqrt{1.5}$.

(c) Determine the actual error of the approximations in part (b).

7. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

   (a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5)-P_2(0.5)|$ using the error formula, and compare it to the actual error.

   (b) Find a bound for the error $|f(x) - P_2(x)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.

   (c) Approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$.

   (d) Find an upper bound for the error in (c) using $\int_0^1 |R_2(x)\,dx|$, and compare the bound to the actual error.

8. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x - 1)\ln x$ about $x_0 = 1$.

   (a) Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5)-P_3(0.5)|$ using the error formula, and compare it to the actual error.

   (b) Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$.

   (c) Approximate $\int_{0.5}^{1.5} f(x)\,dx$ using $\int_{0.5}^{1.5} P_3(x)\,dx$.

   (d) Find an upper bound for the error in (c) using $\int_{0.5}^{1.5} |R_3(x)\,dx|$, and compare the bound to the actual error.

9. Use the error term of a Taylor polynomial to estimate the error involved in using $\sin x \approx x$ to approximate $\sin 1^\circ$.

10. Use a Taylor polynomial about $\pi/4$ to approximate $\cos 42^\circ$ to an accuracy of 10^{-6}.

11. Let $f(x) = e^{x/2}\sin(x/3)$. Use Maple to determine the following.

    (a) The third Maclaurin polynomial $P_3(x)$.

    (b) $f^{(4)}(x)$ and a bound for the error $|f(x) - P_3(x)|$ on $[0, 1]$.

12. Let $f(x) = \ln(x^2 + 2)$. Use Maple to determine the following.

    (a) The Taylor polynomial $P_3(x)$ for f expanded about $x_0 = 1$.

    (b) The maximum error $|f(x) - P_3(x)|$ for $0 \le x \le 1$.

    (c) The Maclaurin polynomial $\tilde{P}_3(x)$ for f.

    (d) The maximum error $|f(x) - \tilde{P}_3(x)|$ for $0 \le x \le 1$.

    (e) Does $P_3(0)$ approximate $f(0)$ better than $\tilde{P}_3(1)$ approximates $f(1)$?

13. The polynomial $P_2(x) = 1 - \frac{1}{2}x^2$ is to be used to approximate $f(x) = \cos x$ in $[-\frac{1}{2}, \frac{1}{2}]$. Find a bound for the maximum error.

14. The nth Taylor polynomial for a function f at x_0 is sometimes referred to as the polynomial of degree at most n that "best" approximates f near x_0.

    (a) Explain why this description is accurate.

    (b) Find the quadratic polynomial that best approximates a function f near $x_0 = 1$ if the tangent line at $x_0 = 1$ has equation $y = 4x - 1$, and if $f''(1) = 6$.

15. The *error function* defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

gives the probability that any one of a series of trials will lie within x units of the mean, assuming that the trials have a normal distribution with mean 0 and standard deviation $\sqrt{2}/2$. This integral cannot be evaluated in terms of elementary functions, so an approximating technique must be used.

    (a) Integrate the Maclaurin series for e^{-t^2} to show that

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

    (b) The error function can also be expressed in the form

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}.$$

    Verify that the two series agree for $k = 1$, 2, 3, and 4. [*Hint:* Use the Maclaurin series for e^{-x^2}.]

    (c) Use the series in part (a) to approximate erf(1) to within 10^{-7}.

    (d) Use the same number of terms used in part (c) to approximate erf(1) with the series in part (b).

    (e) Explain why difficulties occur using the series in part (b) to approximate $\operatorname{erf}(x)$.