2.4 Newton's Method

The Bisection and Secant methods both have geometric representations that use the zero of an approximating line to the graph of a function f to approximate the solution to $f(x) = 0$. The increase in accuracy of the Secant method over the Bisection method is a consequence of the fact that the secant line to the curve better approximates the graph of f than does the line used to generate the approximations in the Bisection method.

The line that *best* approximates the graph of the function at a point on its graph is the tangent line to the graph at that point. Using this line instead of the secant line produces **Newton's method** (also called the *Newton–Raphson method*), the technique we consider in this section.

Suppose that p_0 is an initial approximation to the root p of the equation $f(x) = 0$ and that f' exists in an interval containing all the approximations to p. The slope of the tangent line to the graph of f at the point $(p_0, f(p_0))$ is $f'(p_0)$, so the equation of this tangent line is

$$y - f(p_0) = f'(p_0)(x - p_0).$$

Since this line crosses the x-axis when the y-coordinate of the point on the line is zero, the next approximation, p_1, to p satisfies

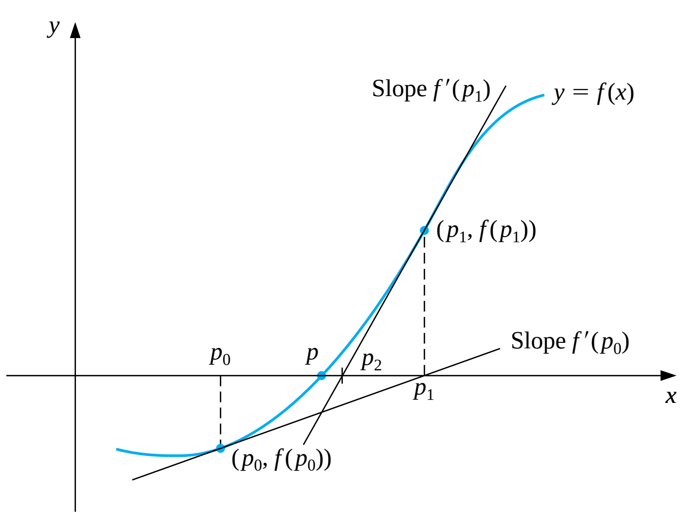
$$0 - f(p_0) = f'(p_0)(p_1 - p_0),$$

which implies that

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)},$$

provided that $f'(p_0) \neq 0$. Subsequent approximations are found for p in a similar manner, as shown in Figure 2.6.

Figure 2.6



> [Newton's Method] The approximation p_{n+1} to a root of $f(x) = 0$ is computed from the approximation p_n using the equation
>
> $$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

EXAMPLE 1 In this example we use Newton's method to approximate the root of the equation $x^3 + 4x^2 - 10 = 0$. Maple is used to find the first iteration of Newton's method with $p_0 = 1$. We define $f(x)$ and compute $f'(x)$ by

```
>f:=x->x^3+4*x^2-10;
>fp:=x->D(f)(x);
>p0:=1;
```

The first iteration of Newton's method gives $p_1 = \frac{16}{11}$, which is obtained with

```
>p1:=p0-f(p0)/fp(p0);
```

A decimal representation of 1.454545455 for p_1 is given by

```
>p1:=evalf(p1);
```

The process can be continued to generate the entries in Table 2.4.

Table 2.4

n	p_n	$f(p_n)$
1	1.45454545454	1.5401953418
2	1.3689004011	0.0607196886
3	1.3652366002	0.0001087706
4	1.3652300134	0.0000000004

We use $p_0 = 1$, $TOL = 0.0005$, and $N_0 = 20$ in the program NEWTON24 to compare the convergence of this method with those applied to this problem previously. The number of iterations needed to solve the problem by Newton's method is less than the number needed for the Secant method, which, in turn, required less than half the iterations needed for the Bisection method. In addition, for Newton's method we have $|p - p_4| \approx 10^{-10}$. □

Newton's method generally produces accurate results in just a few iterations. With the aid of Taylor polynomials we can see why this is true. Suppose p is the solution to $f(x) = 0$ and that f'' exists on an interval containing both p and the approximation p_n. Expanding f in the first Taylor polynomial at p_n and evaluating at $x = p$ gives

$$0 = f(p) = f(p_n) + f'(p_n)(p - p_n) + \frac{f''(\xi)}{2}(p - p_n)^2,$$

where ξ lies between p_n and p. Consequently, if $f'(p_n) \neq 0$, we have

$$p - p_n + \frac{f(p_n)}{f'(p_n)} = -\frac{f''(\xi)}{2f'(p_n)}(p - p_n)^2.$$

Since

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)},$$

this implies that

$$p - p_{n+1} = -\frac{f''(\xi)}{2f'(p_n)}(p - p_n)^2.$$

If a positive constant M exists with $|f''(x)| \leq M$ on an interval about p, and if p_n is within this interval, then

$$|p - p_{n+1}| \leq \frac{M}{2|f'(p_n)|}|p - p_n|^2.$$

The important feature of this inequality is that the error $|p - p_{n+1}|$ of the $(n+1)$st approximation is bounded by approximately the square of the error of the nth approximation, $|p - p_n|$. This implies that Newton's method has the tendency to approximately double the number of digits of accuracy with each successive approximation. Newton's method is not, however, infallible, as the equation in Exercise 12 shows.

EXAMPLE 2 Find an approximation to the solution of the equation $x = 3^{-x}$ that is accurate to within 10^{-8}.

A solution of this equation corresponds to a solution of

$$0 = f(x) = x - 3^{-x}.$$

Since f is continuous with $f(0) = -1$ and $f(1) = \frac{2}{3}$, a solution of the equation lies in the interval (0, 1). We have chosen the initial approximation to be the midpoint of this interval, $p_0 = 0.5$. Succeeding approximations are generated by applying the formula

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n - 3^{-p_n}}{1 + 3^{-p_n}\ln 3}.$$

These approximations are listed in Table 2.5, together with differences between successive approximations. Since Newton's method tends to double the number of decimal places of accuracy with each iteration, it is reasonable to suspect that p_3 is correct at least to the places listed. □

Table 2.5

n	p_n	$\|p_n - p_{n-1}\|$
0	0.500000000	
1	0.547329757	0.047329757
2	0.547808574	0.000478817
3	0.547808622	0.000000048

The success of Newton's method is predicated on the assumption that the derivative of f is nonzero at the approximations to the root p. If f' is continuous, this means that the technique will be satisfactory provided that $f'(p) \neq 0$ and that a sufficiently accurate initial approximation is used. The condition $f'(p) \neq 0$ is not trivial; it is true precisely when p is a *simple root.* A simple root of a function f occurs at p if a function q exists with the property that, for $x \neq p$,

$$f(x) = (x-p)q(x), \qquad \text{where} \qquad \lim_{x \to p} q(x) \neq 0.$$

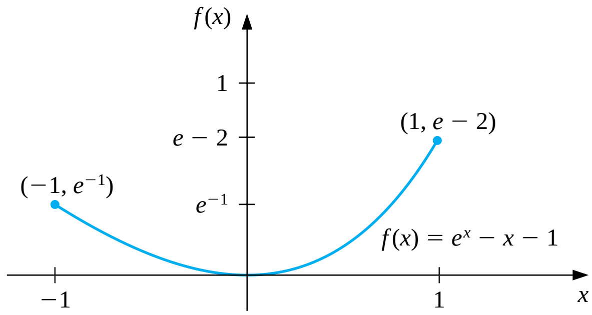
When the root is not simple, Newton's method may converge, but not with the speed we have seen in our previous examples.

EXAMPLE 3 The root $p = 0$ of the equation $f(x) = e^x - x - 1 = 0$ is not simple, since both $f(0) = e^0 - 0 - 1 = 0$ and $f'(0) = e^0 - 1 = 0$. The terms generated by Newton's method with $p_0 = 0$ are shown in Table 2.6 and converge slowly to zero. The graph of f is shown in Figure 2.7. □

Table 2.6

n	p_n	n	p_n
0	1.0	9	2.7750×10^{-3}
1	0.58198	10	1.3881×10^{-3}
2	0.31906	11	6.9411×10^{-4}
3	0.16800	12	3.4703×10^{-4}
4	0.08635	13	1.7416×10^{-4}
5	0.04380	14	8.8041×10^{-5}
6	0.02206		
7	0.01107		
8	0.005545		

Figure 2.7


f(x)
1
e − 2
e⁻¹
(1, e − 2)
(−1, e⁻¹)
f(x) = eˣ − x − 1
−1
1
x

EXERCISE SET 2.4

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used for this problem?

3. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.

   (a) $x^3 - 2x^2 - 5 = 0, \quad \text{on} \quad [1, 4]$

   (b) $x^3 + 3x^2 - 1 = 0, \quad \text{on} \quad [-3, -2]$

   (c) $x - \cos x = 0, \quad \text{on} \quad [0, \pi/2]$

   (d) $x - 0.8 - 0.2 \sin x = 0, \quad \text{on} \quad [0, \pi/2]$

4. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

   (a) $2x \cos 2x - (x - 2)^2 = 0, \quad$ on $[2, 3]$ and $[3, 4]$

   (b) $(x - 2)^2 - \ln x = 0, \quad$ on $[1, 2]$ and $[e, 4]$

   (c) $e^x - 3x^2 = 0, \quad$ on $[0, 1]$ and $[3, 5]$

   (d) $\sin x - e^{-x} = 0, \quad$ on $[0, 1]$, $[3, 4]$, and $[6, 7]$

5. Use Newton's method to find all four solutions of $4x \cos(2x) - (x - 2)^2 = 0$ in $[0, 8]$ accurate to within 10^{-5}.

6. Use Newton's method to find all solutions of $x^2 + 10 \cos x = 0$ accurate to within 10^{-5}.

7. Use Newton's method to approximate the solutions of the following equations to within 10^{-5} in the given intervals. In these problems the convergence will be slower than normal since the roots are not simple roots.

   (a) $x^2 - 2xe^{-x} + e^{-2x} = 0, \quad \text{on} \quad [0, 1]$

   (b) $\cos(x + \sqrt{2}) + x\left(x/2 + \sqrt{2}\right) = 0, \quad \text{on} \quad [-2, -1]$

   (c) $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) + 8^{-x} = 0, \quad \text{on} \quad [0, 1]$

   (d) $e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3, \quad \text{on} \quad [-1, 0]$

8. The numerical method defined by

$$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})},$$

   for $n = 1, 2, \ldots$, can be used instead of Newton's method for equations having multiple roots. Repeat Exercise 7 using this method.

9. Use Newton's method to find an approximation to $\sqrt{3}$ correct to within 10^{-4}, and compare the results to those obtained in Exercise 9 of Sections 2.2 and 2.3.

10. Use Newton's method to find an approximation to $\sqrt[3]{25}$ correct to within 10^{-6}, and compare the results to those obtained in Exercise 10 of Section 2.2 and 2.3.

11. In Exercise 14 of Section 2.3 we found that for $f(x) = \tan \pi x - 6$, the Bisection method on $[0, 0.48]$ converges more quickly than the method of False Position with $p_0 = 0$ and $p_1 = 0.48$. Also, the Secant method with these values of p_0 and p_1 does not give convergence. Apply Newton's method to this problem with (a) $p_0 = 0$, and (b) $p_0 = 0.48$. (c) Explain the reason for any discrepancies.

12. Use Newton's method to determine the first positive solution to the equation $\tan x = x$, and explain why this problem can give difficulties.

13. Use Newton's method to solve the equation
$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x \sin x - \frac{1}{2} \cos 2x, \quad \text{with } p_0 = \frac{\pi}{2}.$$
Iterate using Newton's method until an accuracy of 10^{-5} is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

14. Use Maple to determine how many iterations of Newton's method with $p_0 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within 10^{-100}.

15. Player A will shut out (win by a score of 21–0) player B in a game of racquetball with probability
$$P = \frac{1+p}{2} \left(\frac{p}{1-p+p^2} \right)^{21},$$
where p denotes the probability A will win any specific rally (independent of the server). (See [K,J], p. 267.) Determine, to within 10^{-3}, the minimal value of p that will ensure that A will shut out B in at least half the matches they play.

16. The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ has an infinite number of zeros.

    (a) Determine, within 10^{-6}, the only negative zero.

    (b) Determine, within 10^{-6}, the four smallest positive zeros.

    (c) Determine a reasonable initial approximation to find the nth smallest positive zero of f. [*Hint:* Sketch an approximate graph of f.]

(d) Use part (c) to determine, within 10^{-6}, the 25th smallest positive zero of f.

17. The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation,*

$$A = \frac{P}{i}[(1+i)^n - 1].$$

In this equation A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at \$750,000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

18. Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1+i)^{-n}],$$

known as an *ordinary annuity equation.* In this equation A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of \$135,000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

19. A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{-t/3}$ milligrams per milliliter t hours after A units have been injected. The maximum safe concentration is 1 mg/ml.

    (a) What amount should be injected to reach this maximum safe concentration and when does this maximum occur?

    (b) An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/ml. Determine, to the nearest minute, when this second injection should be given.

    (c) Assuming that the concentration from consecutive injections is additive and that 75% of the amount originally injected is administered in the second injection, when is it time for the third injection?

20. Let $f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$.

    (a) Use the Maple commands `solve` and `fsolve` to try to find all roots of f.

    (b) Plot $f(x)$ to find initial approximations to roots of f.

(c) Use Newton's method to find roots of f to within 10^{-16}.

(d) Find the exact solutions of $f(x) = 0$ algebraically.

2.5 Error Analysis and Accelerating Convergence

In the previous section we found that Newton's method generally converges very rapidly if a sufficiently accurate initial approximation has been found. This rapid speed of convergence is due to the fact that Newton's method produces *quadratically* convergent approximations.

A method that produces a sequence $\{p_n\}$ of approximations that converge to a number p converges **linearly** if, for large values of n, a constant $0 < M < 1$ exists with

$$|p - p_{n+1}| \le M|p - p_n|.$$

The sequence converges **quadratically** if, for large values of n, a constant $0 < M$ exists with

$$|p - p_{n+1}| \le M|p - p_n|^2.$$

The following example illustrates the advantage of quadratic over linear convergence.

EXAMPLE 1 Suppose that $\{p_n\}$ converges linearly to $p = 0$, $\{\hat{p}_n\}$ converges quadratically to $p = 0$, and the constant $M = 0.5$ is the same in each case. Then

$$|p_1| \le M|p_0| \le (0.5) \cdot |p_0| \qquad \text{and} \qquad |\hat{p}_1| \le M|\hat{p}_0|^2 \le (0.5) \cdot |\hat{p}_0|^2.$$

Similarly,

$$|p_2| \le M|p_1| \le 0.5(0.5) \cdot |p_0| = (0.5)^2|p_0|$$

and

$$|\hat{p}_2| \le M|\hat{p}_1|^2 \le 0.5(0.5|\hat{p}_0|^2)^2 = (0.5)^3|q_0|^4.$$

Continuing,

$$|p_3| \le M|p_2| \le 0.5((0.5)^2|p_0|) = (0.5)^3|p_0|$$

and

$$|\hat{p}_3| \le M|\hat{p}_2|^2 \le 0.5((0.5)^3|\hat{p}_0|^4)^2 = (0.5)^7|q_0|^8.$$

In general,

$$|p_n| \le 0.5^n|p_0|, \qquad \text{whereas} \qquad |\hat{p}_n| \le (0.5)^{2^n-1}|\hat{p}_0|^{2^n}$$

for each $n = 1, 2, \ldots$. Table 2.7 illustrates the relative speed of convergence of these error bounds to zero, assuming that $|p_0| = |\hat{p}_0| = 1$.

Table 2.7

n	Linear Convergence Sequence Bound $p_n = (0.5)^n$	Quadratic Convergence Sequence Bound $\hat{p}_n = (0.5)^{2^n-1}$
1	5.0000×10^{-1}	5.0000×10^{-1}
2	2.5000×10^{-1}	1.2500×10^{-1}
3	1.2500×10^{-1}	7.8125×10^{-3}
4	6.2500×10^{-2}	3.0518×10^{-5}
5	3.1250×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

The quadratically convergent sequence is within 10^{-38} of zero by the seventh term. At least 126 terms are needed to ensure this accuracy for the linearly convergent sequence. If $|\hat{p}_0| < 1$, the bound on the sequence $\{\hat{p}_n\}$ will decrease even more rapidly. No significant change will occur, however, if $|p_0| < 1$. □

As illustrated in Example 1, quadratically convergent sequences generally converge much more quickly than those that converge only linearly. However, linearly convergent methods are much more common than those that converge quadratically. **Aitken's Δ^2 method** is a technique that can be used to accelerate the convergence of a sequence that is linearly convergent, regardless of its origin or application.

Suppose $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p. To motivate the construction of a sequence $\{q_n\}$ that converges more rapidly to p than does $\{p_n\}$, let us first assume that the signs of $p_n - p$, $p_{n+1} - p$, and $p_{n+2} - p$ agree and that n is sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

Then

$$(p_{n+1} - p)^2 \approx (p_{n+2} - p)(p_n - p),$$

so

$$p_{n+1}^2 - 2p_{n+1}p + p^2 \approx p_{n+2}p_n - (p_n + p_{n+2})p + p^2$$

and

$$(p_{n+2} + p_n - 2p_{n+1})p \approx p_{n+2}p_n - p_{n+1}^2.$$

Solving for p gives

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Adding and subtracting the terms p_n^2 and $2p_np_{n+1}$ in the numerator and grouping terms appropriately gives

$$\begin{aligned} p &\approx \frac{p_np_{n+2} - 2p_np_{n+1} + p_n^2 - p_{n+1}^2 + 2p_np_{n+1} - p_n^2}{p_{n+2} - 2p_{n+1} + p_n} \\ &= \frac{p_n(p_{n+2} - 2p_{n+1} + p_n) - (p_{n+1}^2 - 2p_np_{n+1} + p_n^2)}{p_{n+2} - 2p_{n+1} + p_n} \\ &= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}. \end{aligned}$$

Aitken's Δ^2 method uses the sequence $\{q_n\}_{n=0}^{\infty}$ defined by this approximation to p.

[Aitken's Δ^2 Method] If $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges linearly to p, and if

$$q_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n},$$

then $\{q_n\}_{n=0}^{\infty}$ also converges to p, and, in general, more rapidly.

EXAMPLE 2 The sequence $\{p_n\}_{n=1}^{\infty}$, where $p_n = \cos(1/n)$, converges linearly to $p = 1$. The first few terms of the sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are given in Table 2.8. It certainly appears that $\{q_n\}_{n=1}^{\infty}$ converges more rapidly to $p = 1$ than does $\{p_n\}_{n=1}^{\infty}$. □

Table 2.8

n	p_n	q_n
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

For a given sequence $\{p_n\}_{n=0}^{\infty}$, the **forward difference**, Δp_n (read "delta p_n"), is defined as

$$\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0.$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2.$$

The definition implies that

$$\Delta^2 p_n = \Delta(p_{n+1} - p_n) = \Delta p_{n+1} - \Delta p_n = (p_{n+2} - p_{n+1}) - (p_{n+1} - p_n),$$

so

$$\Delta^2 p_n = p_{n+2} - 2p_{n+1} + p_n.$$

Thus, the formula for q_n given in Aitken's Δ^2 method can be written as

$$q_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}, \quad \text{for all } n \geq 0.$$

The sequence $\{q_n\}_{n=1}^{\infty}$ converges to p more rapidly than does the original sequence $\{p_n\}_{n=0}^{\infty}$ in the following sense:

[Aitken's Δ^2 Convergence] If $\{p_n\}$ is a sequence that converges linearly to the limit p and $(p_n - p)(p_{n+1} - p) > 0$ for large values of n, and

$$q_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}, \qquad \text{then} \qquad \lim_{n\to\infty} \frac{q_n - p}{p_n - p} = 0.$$

We will find occasion to apply this acceleration technique at various times in our study of approximation methods.

EXERCISE SET 2.5

1. The following sequences are linearly convergent. Generate the first five terms of the sequence $\{q_n\}$ using Aitken's Δ^2 method.

   (a) $p_0 = 0.5, \quad p_n = (2 - e^{p_{n-1}} + p_{n-1}^2)/3, \quad \text{for } \; n \geq 1$

   (b) $p_0 = 0.75, \quad p_n = (e^{p_{n-1}}/3)^{1/2}, \quad \text{for } n \geq 1$

   (c) $p_0 = 0.5, \quad p_n = 3^{-p_{n-1}}, \quad \text{for } n \geq 1$

   (d) $p_0 = 0.5, \quad p_n = \cos p_{n-1}, \quad \text{for } n \geq 1$

2. Newton's method does not converge quadratically for the following problems. Accelerate the convergence using the Aitken's Δ^2 method. Iterate until $|q_n - q_{n-1}| < 10^{-4}$.

   (a) $x^2 - 2xe^{-x} + e^{-2x} = 0, \quad [0, 1]$

   (b) $\cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0, \quad [-2, -1]$

   (c) $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) - 8^{-x} = 0, \quad [0, 1]$

   (d) $e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3 = 0, \quad [-1, 0]$

3. Consider the function $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$. Use Newton's method with $p_0 = 0$ to approximate a zero of f. Generate terms until $|p_{n+1} - p_n| < 0.0002$. Construct the Aitken's Δ^2 sequence $\{q_n\}$. Is the convergence improved?

4. Repeat Exercise 3 with the constants in $f(x)$ replaced by their four-digit approximations, that is, with $f(x) = e^{6x} + 1.441e^{2x} - 2.079e^{4x} - 0.3330$, and compare the solutions to the results in Exercise 3.

5. (i) Show that the following sequences $\{p_n\}$ converge linearly to $p = 0$. (ii) How large must n be before $|p_n - p| \leq 5 \times 10^{-2}$? (iii) Use Aitken's Δ^2 method to generate a sequence $\{q_n\}$ until $|q_n - p| \leq 5 \times 10^{-2}$.

   (a) $p_n = \dfrac{1}{n}$, for $n \geq 1$

   (b) $p_n = \dfrac{1}{n^2}$, for $n \geq 1$

6. (a) Show that for any positive integer k, the sequence defined by $p_n = 1/n^k$ converges linearly to $p = 0$.

   (b) For each pair of integers k and m, determine a number N for which $1/N^k < 10^{-m}$.

7. (a) Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to zero.

(b) Show that the sequence $p_n = 10^{-n^k}$ does not converge to zero quadratically, regardless of the size of the exponent $k > 1$.

8. A sequence $\{p_n\}$ is said to be **superlinearly convergent** to p if a sequence $\{c_n\}$ converging to zero exists with

$$|p_{n+1} - p| \leq c_n |p_n - p|.$$

(a) Show that if $\{p_n\}$ is superlinearly convergent to p, then $\{p_n\}$ is linearly convergent to p.

(b) Show that $p_n = 1/n^n$ is superlinearly convergent to zero but is not quadratically convergent to zero.