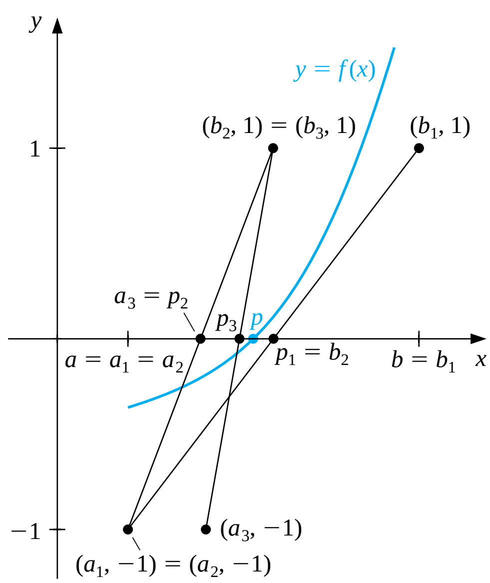
2.3 The Secant Method

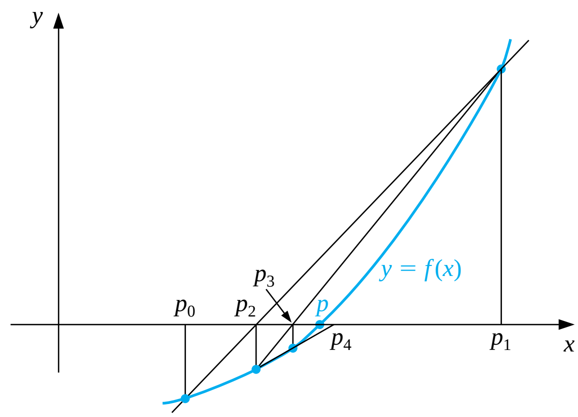
Although the Bisection method always converges, the speed of convergence is usually too slow for general use. Figure 2.3 gives a graphical interpretation of the Bisection method that can be used to discover how improvements on this technique can be derived. It shows the graph of a continuous function that is negative at a_1 and positive at b_1. The first approximation p_1 to the root p is found by drawing the line joining the points $(a_1, \text{sgn}(f(a_1))) = (a_1, -1)$ and $(b_1, \text{sgn}(f(b_1))) = (b_1, 1)$ and letting p_1 be the point where this line intersects the x-axis. In essence, the line joining $(a_1, -1)$ and $(b_1, 1)$ has been used to approximate the graph of f on the interval $[a_1, b_1]$. Successive approximations apply this same process on subintervals of $[a_1, b_1]$, $[a_2, b_2]$, and so on. Notice that the Bisection method uses no information about the function f except the fact that $f(x)$ is positive and negative at certain values of x.

Figure 2.3



Suppose that in the initial step we know that $|f(a_1)| < |f(b_1)|$. Then we would expect the root p to be closer to a_1 than to b_1. Alternatively, if $|f(b_1)| < |f(a_1)|$, p is likely to be closer to b_1 than to a_1. Instead of choosing the intersection of the line through $(a_1, \text{sgn}(f(a_1))) = (a_1, -1)$ and $(b_1, \text{sgn}(f(b_1))) = (b_1, 1)$ as the approximation to the root p, the *Secant method* chooses the x-intercept of the secant line to the curve, the line through $(a_1, f(a_1))$ and $(b_1, f(b_1))$. This places the approximation closer to the endpoint of the interval for which f has smaller absolute value, as shown in Figure 2.4.

Figure 2.4



The sequence of approximations generated by the Secant method is started by setting $p_0 = a$ and $p_1 = b$. The equation of the secant line through $(p_0, f(p_0))$ and $(p_1, f(p_1))$ is

$$y = f(p_1) + \frac{f(p_1) - f(p_0)}{p_1 - p_0}(x - p_1).$$

The x-intercept $(p_2, 0)$ of this line satisfies

$$0 = f(p_1) + \frac{f(p_1) - f(p_0)}{p_1 - p_0}(p_2 - p_1)$$

and solving for p_2 gives

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}.$$

[Secant Method] The approximation p_{n+1}, for $n > 1$, to a root of $f(x) = 0$ is computed from the approximations p_n and p_{n-1} using the equation

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}.$$

The Secant method does not have the root-bracketing property of the Bisection method. As a consequence, the method does not always converge, but when it does converge, it generally does so much faster than the Bisection method.

We use two stopping conditions in the Secant method. First, we assume that p_n is sufficiently accurate when $|p_n - p_{n-1}|$ is within a given tolerance. Also, a safeguard exit based upon a maximum number of iterations is given in case the method fails to converge as expected.

The iteration equation should *not* be simplified algebraically to

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})} = \frac{f(p_{n-1})p_n - f(p_n)p_{n-1}}{f(p_{n-1}) - f(p_n)}.$$

Although this is algebraically equivalent to the iteration equation, it could increase the significance of rounding error if the nearly equal numbers $f(p_{n-1})p_n$ and $f(p_n)p_{n-1}$ are subtracted.

EXAMPLE 1 In this example we will approximate a root of the equation $x^3 + 4x^2 - 10 = 0$. To use Maple we first define the function $f(x)$ and the numbers p_0 and p_1 with the commands

```
>f:=x->x^3+4*x^2-10;
>p0:=1; p1:=2;
```

The values of $f(p_0) = -5$ and $f(p_1) = 14$ are computed by

```
>fp0:=f(p0); fp1:=f(p1);
```

and the first secant approximation, $p_2 = \frac{24}{19}$, by

```
>p2:=p1-fp1*(p1-p0)/(fp1-fp0);
```

The next command forces a floating-point representation for p_2 instead of an exact rational representation.

```
>p2:=evalf(p2);
```

We compute $f(p_2) = -1.602274379$ and continue to compute $p_3 = 1.338827839$ by

```
>fp2:=f(p2);
>p3:=p2-fp2*(p2-p1)/(fp2-fp1);
```

The program SECANT22 with inputs $p_0 = 1$, $p_1 = 2$, $TOL = 0.0005$, and $N_0 = 20$ produces the results in Table 2.2. About half the number of iterations are needed, compared to the Bisection method in Example 1 of Section 2.2. Further, $|p - p_6| = |1.3652300134 - 1.3652300011| < 1.3 \times 10^{-8}$ is much smaller than the tolerance 0.0005. □

Table 2.2

n	p_n	$f(p_n)$
2	1.2631578947	−1.6022743840
3	1.3388278388	−0.4303647480
4	1.3666163947	0.0229094308
5	1.3652119026	−0.0002990679
6	1.3652300011	−0.0000002032

There are other reasonable choices for generating a sequence of approximations based on the intersection of an approximating line and the x-axis. The **method of False Position** (or *Regula Falsi*)is a hybrid bisection-secant method that constructs approximating lines similar to those of the Secant method but always brackets the root in the manner of the Bisection method. As with the Bisection method, the method of False Position requires that an initial interval $[a, b]$ first be found, with $f(a)$ and $f(b)$ of opposite sign. With $a_1 = a$ and $b_1 = b$, the approximation, p_2, is given by

$$p_2 = a_1 - \frac{f(a_1)(b_1 - a_1)}{f(b_1) - f(a_1)}.$$

If $f(p_2)$ and $f(a_1)$ have the same sign, then set $a_2 = p_2$ and $b_2 = b_1$. Alternatively, if $f(p_2)$ and $f(b_1)$ have the same sign, set $a_2 = a_1$ and $b_2 = p_2$. (See Figure 2.5.)

[Method of False Position] An interval $[a_{n+1}, b_{n+1}]$, for $n > 1$, containing an approximation to a root of $f(x) = 0$ is found from an interval $[a_n, b_n]$ containing the root by first computing

$$p_{n+1} = a_n - \frac{f(a_n)(b_n - a_n)}{f(b_n) - f(a_n)}.$$

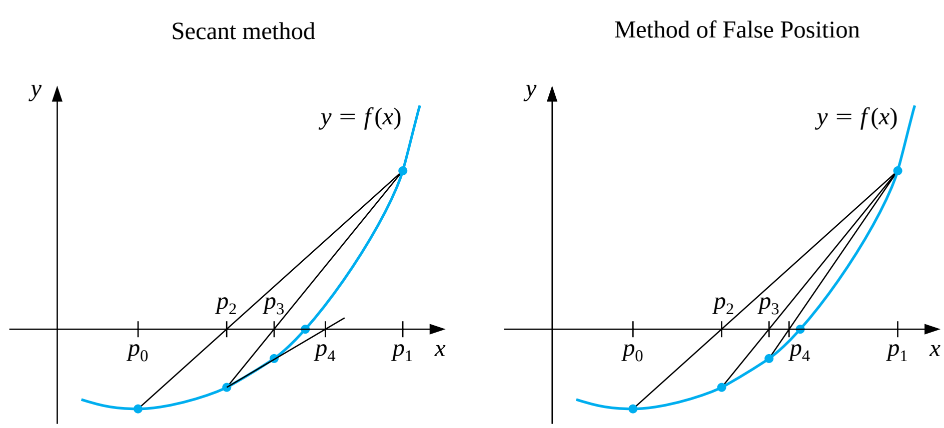
Then set

$$a_{n+1} = a_n \quad \text{and} \quad b_{n+1} = p_{n+1} \quad \text{if} \quad f(a_n)f(p_{n+1}) < 0,$$

and

$$a_{n+1} = p_{n+1} \quad \text{and} \quad b_{n+1} = b_n \quad \text{otherwise.}$$

Figure 2.5



Although the method of False Position may appear superior to the Secant method, it generally converges more slowly, as the results in Table 2.3 indicate for the problem we considered in Example 1. In fact, the method of False Position can converge even more slowly than the Bisection method (as the problem given in Exercise 14 shows), although this is not usually the case. The program FALPOS23 implements the method of False Position.

Table 2.3

n	a_n	b_n	p_{n+1}	$f(p_{n+1})$
1	1.00000000	2.00000000	1.26315789	−1.60227438
2	1.26315789	2.00000000	1.33882784	−0.43036475
3	1.33882784	2.00000000	1.35854634	−0.11000879
4	1.35854634	2.00000000	1.36354744	−0.02776209
5	1.36354744	2.00000000	1.36480703	−0.00698342
6	1.36480703	2.00000000	1.36512372	−0.00175521
7	1.36512372	2.00000000	1.36520330	−0.00044106

EXERCISE SET 2.3

1. Let $f(x) = x^2 - 6$, $p_0 = 3$, and $p_1 = 2$. Find p_3 using each method.

   (a) Secant method
   (b) method of False Position

2. Let $f(x) = -x^3 - \cos x$, $p_0 = -1$, and $p_1 = 0$. Find p_3 using each method.

   (a) Secant method
   (b) method of False Position

3. Use the Secant method to find solutions accurate to within 10^{-4} for the following problems.

   (a) $x^3 - 2x^2 - 5 = 0$, on $[1, 4]$
   (b) $x^3 + 3x^2 - 1 = 0$, on $[-3, -2]$
   (c) $x - \cos x = 0$, on $[0, \pi/2]$
   (d) $x - 0.8 - 0.2 \sin x = 0$, on $[0, \pi/2]$

4. Use the Secant method to find solutions accurate to within 10^{-5} for the following problems.

   (a) $2x \cos 2x - (x - 2)^2 = 0$ on $[2, 3]$ and on $[3, 4]$
   (b) $(x - 2)^2 - \ln x = 0$ on $[1, 2]$ and on $[e, 4]$
   (c) $e^x - 3x^2 = 0$ on $[0, 1]$ and on $[3, 5]$
   (d) $\sin x - e^{-x} = 0$ on $[0, 1]$, on $[3, 4]$ and on $[6, 7]$

5. Repeat Exercise 3 using the method of False Position.

6. Repeat Exercise 4 using the method of False Position.

7. Use the Secant method to find all four solutions of $4x \cos(2x) - (x - 2)^2 = 0$ in $[0, 8]$ accurate to within 10^{-5}.

8. Use the Secant method to find all solutions of $x^2 + 10 \cos x = 0$ accurate to within 10^{-5}.

9. Use the Secant method to find an approximation to $\sqrt{3}$ correct to within 10^{-4}, and compare the results to those obtained in Exercise 9 of Section 2.2.

10. Use the Secant method to find an approximation to $\sqrt[3]{25}$ correct to within 10^{-6}, and compare the results to those obtained in Exercise 10 of Section 2.2.

11. Approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$. [*Hint:* Minimize $[d(x)]^2$, where $d(x)$ represents the distance from (x, x^2) to $(1, 0)$.]

12. Approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = 1/x$ that is closest to (2, 1).

13. The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using each method.

(a) method of False Position (b) Secant method

14. The function $f(x) = \tan \pi x - 6$ has a zero at $(1/\pi) \arctan 6 \approx 0.447431543$. Let $p_0 = 0$ and $p_1 = 0.48$ and use 10 iterations of each of the following methods to approximate this root. Which method is most successful and why?

(a) Bisection method

(b) method of False Position

(c) Secant method

15. Use Maple to determine how many iterations of the Secant method with $p_0 = \frac{1}{2}$ and $p_1 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within 10^{-100}.

16. The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within 10^{-4}.

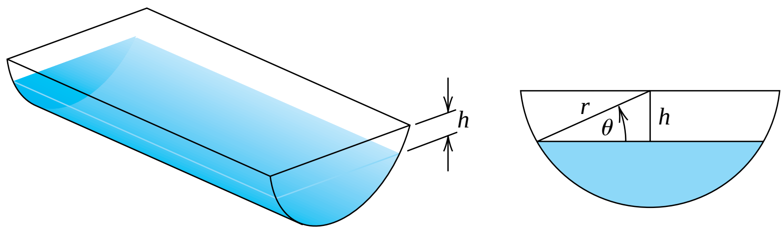
17. A trough of length L has a cross section in the shape of a semicircle with radius r. (See the accompanying figure.) When filled with water to within a distance h of the top, the volume, V, of water is

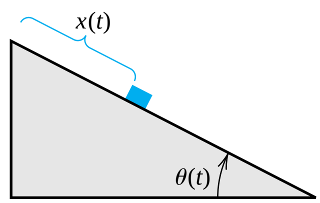
$$V = L\left[0.5\pi r^2 - r^2 \arcsin\left(\frac{h}{r}\right) - h(r^2 - h^2)^{1/2}\right]$$

Suppose $L = 10$ ft, $r = 1$ ft, and $V = 12.4$ ft^3. Find the depth of water in the trough to within 0.01 ft.

18. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$





At the end of t seconds, the position of the object is given by

$$x(t) = \frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = -32.17$ ft/s^2.