## The Method of False Position

Each successive pair of approximations in the Bisection method brackets a root $p$ of the equation; that is, for each positive integer $n$, a root lies between $a_n$ and $b_n$. This implies that, for each $n$, the Bisection method iterations satisfy

$$|p_n - p| < \frac{1}{2}|a_n - b_n|,$$

which provides an easily calculated error bound for the approximations.

The term *Regula Falsi*, literally a false rule or false position, refers to a technique that uses results that are known to be false, but in some specific manner, to obtain convergence to a true result. False position problems can be found on the Rhind papyrus, which dates from about 1650 B.C.E.

Root bracketing is not guaranteed for either Newton's method or the Secant method. In Example 1, Newton's method was applied to $f(x) = \cos x - x$, and an approximate root was found to be 0.7390851332. Table 2.5 shows that this root is not bracketed by either $p_0$ and $p_1$ or $p_1$ and $p_2$. The Secant method approximations for this problem are also given in Table 2.5. In this case the initial approximations $p_0$ and $p_1$ bracket the root, but the pair of approximations $p_3$ and $p_4$ fail to do so.

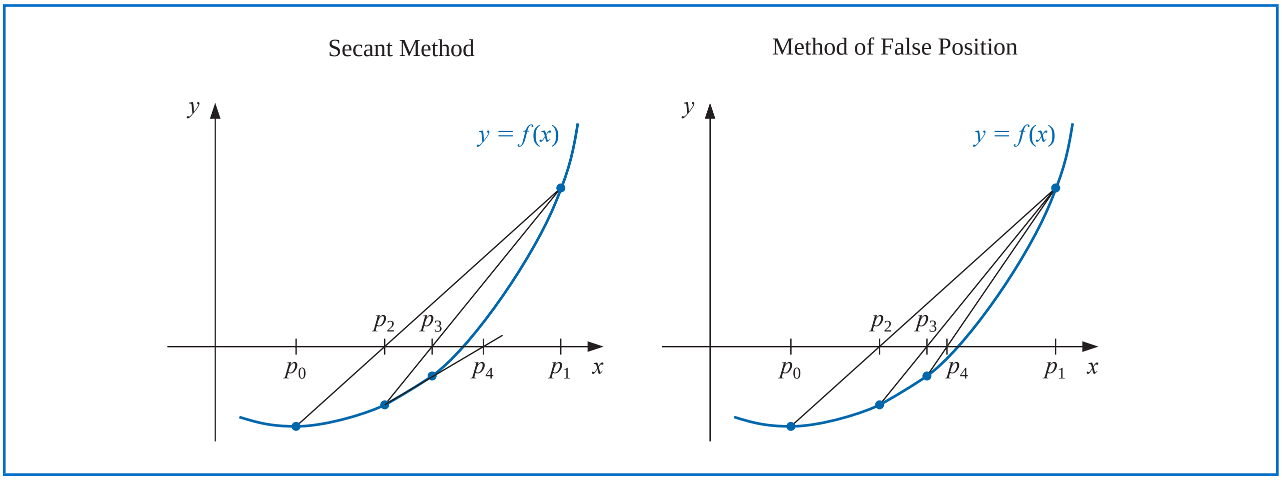
The **method of False Position** (also called *Regula Falsi*) generates approximations in the same manner as the Secant method, but it includes a test to ensure that the root is always bracketed between successive iterations. Although it is not a method we generally recommend, it illustrates how bracketing can be incorporated.

First choose initial approximations $p_0$ and $p_1$ with $f(p_0) \cdot f(p_1) < 0$. The approximation $p_2$ is chosen in the same manner as in the Secant method, as the $x$-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. To decide which secant line to use to compute $p_3$, consider $f(p_2) \cdot f(p_1)$, or more correctly $\operatorname{sgn} f(p_2) \cdot \operatorname{sgn} f(p_1)$.

- If $\operatorname{sgn} f(p_2) \cdot \operatorname{sgn} f(p_1) < 0$, then $p_1$ and $p_2$ bracket a root. Choose $p_3$ as the $x$-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$.
- If not, choose $p_3$ as the $x$-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on $p_0$ and $p_1$.

In a similar manner, once $p_3$ is found, the sign of $f(p_3) \cdot f(p_2)$ determines whether we use $p_2$ and $p_3$ or $p_3$ and $p_1$ to compute $p_4$. In the latter case a relabeling of $p_2$ and $p_1$ is performed. The relabeling ensures that the root is bracketed between successive iterations. The process is described in Algorithm 2.5, and Figure 2.11 shows how the iterations can differ from those of the Secant method. In this illustration, the first three approximations are the same, but the fourth approximations differ.

**Figure 2.11**



ALGORITHM 2.5

## False Position

To find a solution to $f(x) = 0$ given the continuous function $f$ on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:

INPUT initial approximations $p_0, p_1$; tolerance $TOL$; maximum number of iterations $N_0$.

OUTPUT approximate solution $p$ or message of failure.

*Step 1* Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

*Step 2* While $i \le N_0$ do Steps 3–7.

*Step 3* Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* $p_i$.)

*Step 4* If $|p - p_1| < TOL$ then
OUTPUT ($p$); (*The procedure was successful.*)
STOP.

*Step 5* Set $i = i + 1$;
$q = f(p)$.

*Step 6* If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.

*Step 7* Set $p_1 = p$;
$q_1 = q$.

*Step 8* OUTPUT ('Method failed after $N_0$ iterations, $N_0$ =', $N_0$);
(*The procedure unsuccessful.*)
STOP. ■

**Example 3** Use the method of False Position to find a solution to $x = \cos x$, and compare the approximations with those given in Example 1 which applied fixed-point iteration and Newton's method, and to those found in Example 2 which applied the Secant method.

***Solution*** To make a reasonable comparison we will use the same initial approximations as in the Secant method, that is, $p_0 = 0.5$ and $p_1 = \pi/4$. Table 2.6 shows the results of the method of False Position applied to $f(x) = \cos x - x$ together with those we obtained using the Secant and Newton's methods. Notice that the False Position and Secant approximations agree through $p_3$ and that the method of False Position requires an additional iteration to obtain the same accuracy as the Secant method. ■

**Table 2.6**

| | False Position | Secant | Newton |
|---|---|---|---|
| $n$ | $p_n$ | $p_n$ | $p_n$ |
| 0 | 0.5 | 0.5 | 0.7853981635 |
| 1 | 0.7853981635 | 0.7853981635 | 0.7395361337 |
| 2 | 0.7363841388 | 0.7363841388 | 0.7390851781 |
| 3 | 0.7390581392 | 0.7390581392 | 0.7390851332 |
| 4 | 0.7390848638 | 0.7390851493 | 0.7390851332 |
| 5 | 0.7390851305 | 0.7390851332 | |
| 6 | 0.7390851332 | | |

The added insurance of the method of False Position commonly requires more calculation than the Secant method, just as the simplification that the Secant method provides over Newton's method usually comes at the expense of additional iterations. Further examples of the positive and negative features of these methods can be seen by working Exercises 17 and 18.

Maple has Newton's method, the Secant method, and the method of False Position implemented in its *NumericalAnalysis* package. The options that were available for the Bisection method are also available for these techniques. For example, to generate the results in Tables 2.4, 2.5, and 2.6 we could use the commands

$with(Student[NumericalAnalysis])$

$f := \cos(x) - x$

$Newton\left(f, x = \frac{\pi}{4.0}, tolerance = 10^{-8}, output = sequence, maxiterations = 20\right)$

$Secant\left(f, x = \left[0.5, \frac{\pi}{4.0}\right], tolerance = 10^{-8}, output = sequence, maxiterations = 20\right)$

and

$FalsePosition\left(f, x = \left[0.5, \frac{\pi}{4.0}\right], tolerance = 10^{-8}, output = sequence, maxiterations = 20\right)$

## EXERCISE SET 2.3

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find $p_2$.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find $p_2$. Could $p_0 = 0$ be used?
3. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find $p_3$.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.
   - **c.** Which of **a.** or **b.** is closer to $\sqrt{6}$?
4. Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find $p_3$.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.
5. Use Newton's method to find solutions accurate to within $10^{-4}$ for the following problems.
   - **a.** $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$
   - **b.** $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$
   - **c.** $x - \cos x = 0, \quad [0, \pi/2]$
   - **d.** $x - 0.8 - 0.2 \sin x = 0, \quad [0, \pi/2]$
6. Use Newton's method to find solutions accurate to within $10^{-5}$ for the following problems.
   - **a.** $e^x + 2^{-x} + 2\cos x - 6 = 0 \quad$ for $1 \le x \le 2$
   - **b.** $\ln(x - 1) + \cos(x - 1) = 0 \quad$ for $1.3 \le x \le 2$
   - **c.** $2x \cos 2x - (x - 2)^2 = 0 \quad$ for $2 \le x \le 3$ and $3 \le x \le 4$
   - **d.** $(x - 2)^2 - \ln x = 0 \quad$ for $1 \le x \le 2$ and $e \le x \le 4$
   - **e.** $e^x - 3x^2 = 0 \quad$ for $0 \le x \le 1$ and $3 \le x \le 5$
   - **f.** $\sin x - e^{-x} = 0 \quad$ for $0 \le x \le 1$ $3 \le x \le 4$ and $6 \le x \le 7$
7. Repeat Exercise 5 using the Secant method.
8. Repeat Exercise 6 using the Secant method.
9. Repeat Exercise 5 using the method of False Position.
10. Repeat Exercise 6 using the method of False Position.
11. Use all three methods in this Section to find solutions to within $10^{-5}$ for the following problems.
    - **a.** $3xe^x = 0 \quad$ for $1 \le x \le 2$
    - **b.** $2x + 3\cos x - e^x = 0 \quad$ for $0 \le x \le 1$

**12.** Use all three methods in this Section to find solutions to within $10^{-7}$ for the following problems.

**a.** $x^2 - 4x + 4 - \ln x = 0$ for $1 \leq x \leq 2$ and for $2 \leq x \leq 4$

**b.** $x + 1 - 2\sin \pi x = 0$ for $0 \leq x \leq 1/2$ and for $1/2 \leq x \leq 1$

**13.** Use Newton's method to approximate, to within $10^{-4}$, the value of $x$ that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$. [*Hint:* Minimize $[d(x)]^2$, where $d(x)$ represents the distance from $(x, x^2)$ to $(1, 0)$.]

**14.** Use Newton's method to approximate, to within $10^{-4}$, the value of $x$ that produces the point on the graph of $y = 1/x$ that is closest to $(2, 1)$.

**15.** The following describes Newton's method graphically: Suppose that $f'(x)$ exists on $[a, b]$ and that $f'(x) \neq 0$ on $[a, b]$. Further, suppose there exists one $p \in [a, b]$ such that $f(p) = 0$, and let $p_0 \in [a, b]$ be arbitrary. Let $p_1$ be the point at which the tangent line to $f$ at $(p_0, f(p_0))$ crosses the $x$-axis. For each $n \geq 1$, let $p_n$ be the $x$-intercept of the line tangent to $f$ at $(p_{n-1}, f(p_{n-1}))$. Derive the formula describing this method.

**16.** Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x, \quad \text{with } p_0 = \frac{\pi}{2}.$$

Iterate using Newton's method until an accuracy of $10^{-5}$ is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

**17.** The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within $10^{-6}$ using the

**a.** Method of False Position

**b.** Secant method

**c.** Newton's method

Use the endpoints of each interval as the initial approximations in (a) and (b) and the midpoints as the initial approximation in (c).

**18.** The function $f(x) = \tan \pi x - 6$ has a zero at $(1/\pi) \arctan 6 \approx 0.447431543$. Let $p_0 = 0$ and $p_1 = 0.48$, and use ten iterations of each of the following methods to approximate this root. Which method is most successful and why?

**a.** Bisection method

**b.** Method of False Position

**c.** Secant method

**19.** The iteration equation for the Secant method can be written in the simpler form

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

Explain why, in general, this iteration equation is likely to be less accurate than the one given in Algorithm 2.4.

**20.** The equation $x^2 - 10\cos x = 0$ has two solutions, $\pm 1.3793646$. Use Newton's method to approximate the solutions to within $10^{-5}$ with the following values of $p_0$.

**a.** $p_0 = -100$ **b.** $p_0 = -50$ **c.** $p_0 = -25$

**d.** $p_0 = 25$ **e.** $p_0 = 50$ **f.** $p_0 = 100$

**21.** The equation $4x^2 - e^x - e^{-x} = 0$ has two positive solutions $x_1$ and $x_2$. Use Newton's method to approximate the solution to within $10^{-5}$ with the following values of $p_0$.

| | | |
|---|---|---|
| **a.** $p_0 = -10$ | **b.** $p_0 = -5$ | **c.** $p_0 = -3$ |
| **d.** $p_0 = -1$ | **e.** $p_0 = 0$ | **f.** $p_0 = 1$ |
| **g.** $p_0 = 3$ | **h.** $p_0 = 5$ | **i.** $p_0 = 10$ |

**22.** Use Maple to determine how many iterations of Newton's method with $p_0 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within $10^{-100}$.

**23.** The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ has an infinite number of zeros.

**a.** Determine, within $10^{-6}$, the only negative zero.

**b.** Determine, within $10^{-6}$, the four smallest positive zeros.

**c.** Determine a reasonable initial approximation to find the $n$th smallest positive zero of $f$. [*Hint:* Sketch an approximate graph of $f$.]

**d.** Use part (c) to determine, within $10^{-6}$, the 25th smallest positive zero of $f$.

**24.** Find an approximation for $\lambda$, accurate to within $10^{-4}$, for the population equation

$$1{,}564{,}000 = 1{,}000{,}000 e^{\lambda} + \frac{435{,}000}{\lambda}(e^{\lambda} - 1),$$

discussed in the introduction to this chapter. Use this value to predict the population at the end of the second year, assuming that the immigration rate during this year remains at 435,000 individuals per year.

**25.** The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within $10^{-4}$.

**26.** The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation*,

$$A = \frac{P}{i}[(1 + i)^n - 1].$$

In this equation, $A$ is the amount in the account, $P$ is the amount regularly deposited, and $i$ is the rate of interest per period for the $n$ deposit periods. An engineer would like to have a savings account valued at \$750,000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

**27.** Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1 + i)^{-n}],$$

known as an *ordinary annuity equation.* In this equation, $A$ is the amount of the mortgage, $P$ is the amount of each payment, and $i$ is the interest rate per period for the $n$ payment periods. Suppose that a 30-year home mortgage in the amount of \$135,000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

**28.** A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{-t/3}$ milligrams per milliliter, $t$ hours after $A$ units have been injected. The maximum safe concentration is 1 mg/mL.

**a.** What amount should be injected to reach this maximum safe concentration, and when does this maximum occur?

**b.** An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/mL. Determine, to the nearest minute, when this second injection should be given.

**c.** Assume that the concentration from consecutive injections is additive and that 75% of the amount originally injected is administered in the second injection. When is it time for the third injection?

**29.** Let $f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$.

**a.** Use the Maple commands *solve* and *fsolve* to try to find all roots of $f$.

**b.** Plot $f(x)$ to find initial approximations to roots of $f$.

**c.** Use Newton's method to find roots of $f$ to within $10^{-16}$.

**d.** Find the exact solutions of $f(x) = 0$ without using Maple.

**30.** Repeat Exercise 29 using $f(x) = 2^{x^2} - 3 \cdot 7^{x+1}$.

**31.** The logistic population growth model is described by an equation of the form

$$P(t) = \frac{P_L}{1 - ce^{-kt}},$$

where $P_L, c$, and $k > 0$ are constants, and $P(t)$ is the population at time $t$. $P_L$ represents the limiting value of the population since $\lim_{t\to\infty} P(t) = P_L$. Use the census data for the years 1950, 1960, and 1970 listed in the table on page 105 to determine the constants $P_L, c$, and $k$ for a logistic growth model. Use the logistic model to predict the population of the United States in 1980 and in 2010, assuming $t = 0$ at 1950. Compare the 1980 prediction to the actual value.

**32.** The Gompertz population growth model is described by

$$P(t) = P_L e^{-ce^{-kt}},$$

where $P_L, c$, and $k > 0$ are constants, and $P(t)$ is the population at time $t$. Repeat Exercise 31 using the Gompertz growth model in place of the logistic model.

**33.** Player A will shut out (win by a score of 21–0) player B in a game of racquetball with probability

$$P = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21},$$

where $p$ denotes the probability A will win any specific rally (independent of the server). (See [Keller, J], p. 267.) Determine, to within $10^{-3}$, the minimal value of $p$ that will ensure that A will shut out B in at least half the matches they play.

**34.** In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called *hang-up failure* and occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called *nose-in failure* and occurs when the vehicle descends into a ditch and its nose touches the ground.

The accompanying figure, adapted from [Bek], shows the components associated with the nose-in failure of a vehicle. In that reference it is shown that the maximum angle $\alpha$ that can be negotiated by a vehicle when $\beta$ is the maximum angle at which hang-up failure does *not* occur satisfies the equation

$$A \sin\alpha \cos\alpha + B \sin^2\alpha - C\cos\alpha - E\sin\alpha = 0,$$

where

$$A = l\sin\beta_1, \quad B = l\cos\beta_1, \quad C = (h + 0.5D)\sin\beta_1 - 0.5D\tan\beta_1,$$

$$\text{and} \quad E = (h + 0.5D)\cos\beta_1 - 0.5D.$$

**a.** It is stated that when $l = 89$ in., $h = 49$ in., $D = 55$ in., and $\beta_1 = 11.5°$, angle $\alpha$ is approximately 33°. Verify this result.

**b.** Find $\alpha$ for the situation when $l, h$, and $\beta_1$ are the same as in part (a) but $D = 30$ in.