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محاضرات الاحصاء ١

مدرس المادة : الاستاذ المساعد الدكتور

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Transformation Method of one dimensional

Theorem. Let X be a continuous random variable with probability density function f(X). Let $y = T(x)$ be an increasing (or decreasing) functon .Then the density function of the random density function of the random variable $Y = T(x)$ is given by

$$g(y) = \left|\frac{dx}{dy}\right| f(W(y))$$

Where $x = W(y)$ is the inverse function of T(x).

Proof:- suppose $y = T(x)$ is an increasing function . The distribution function G(y) of Y is given by

$$G(y) = P(Y \leq y)$$

$$= P(T(x) \leq y)$$

$$= P(X \leq W(y))$$

$$= \int_{-\infty}^{W(y)} f(x)dx.$$

Then, differentiating we get the density function of Y , which is

$$g(y) = \frac{dG(y)}{dy}$$

$$= \frac{d}{dy}\left(\int_{-\infty}^{W(y)} f(x)dx\right)$$

$$= f(W(y))\frac{dW(y)}{dy}$$

$$= f\big(W(y)\big)\frac{dx}{dy} \qquad \big(since \quad x = W(y)\big).$$

On the other hand, if $y = T(x)$ is a decreasing function, then the distribution function of Y is given by

$$
\begin{aligned}
G(y) = \; & P(Y \leq y) \\
& = P(T(x) \leq y) \\
& = P\big(X \geq W(y)\big) \qquad (\, since\, T(x)\, is\, decreasing\,) \\
& = 1 - P(X \leq (W(y)) \\
& = 1 - \int_{-\infty}^{W(y)} f(x)dx.
\end{aligned}
$$

As before, differentiating we get the density function of Y, which is

$$
\begin{aligned}
g(y) = \; & \frac{dG(y)}{dy} \\
& = \frac{d}{dy}\left(1 - \int_{-\infty}^{W(y)} f(x)dx\right) \\
& = -f(W(y))\frac{dW(y)}{dy} \\
& = -f(W(y))\frac{dx}{dy} \qquad \big(since \quad x = W(y)\big).
\end{aligned}
$$

Hence, combining both the cases, we get

$$g(y) = \left|\frac{dx}{dy}\right| f(W(y))$$

And the proof of the theorem is now complete .

Example: $Let\ f(x) = \frac{1}{x} \qquad x \geq 1\ Find\ the\ p.d.f\ of\ Y = x$

Solution: $f(x) = \begin{cases} \frac{1}{x} & for\ \ x \geq 1 \\ 0 & o.w \end{cases}$

$g(y) = f[\omega(y)].|J|$

$y = x \Longrightarrow x = y$

$$f[\omega(y)] = \begin{cases} \frac{1}{y} & for\ y \geq 1 \\ 0 & o.w \end{cases}$$

$$|J| = \left|\frac{dx}{dy}\right| = 1$$

$$g(y) = \begin{cases} \frac{1}{y} & for\ y \geq 1 \\ 0 & o.w \end{cases}$$

Example: If $x \sim f(x) = 2x\ for\ 0 < x < 1$. Find the distribution of $Y = 4x^2$.

Solution:-

$$f(x) = \begin{cases} 2x & for\ 0 < x < 1 \\ 0 & o.w \end{cases}$$

$g(y) = f[\omega(y)].|J|$

$[y = 4x^2]\ \div 4$

$$\Rightarrow x^2 = \frac{y}{4}$$

$$\Rightarrow x = \frac{1}{2}\sqrt{y}$$

$$f[\omega(y)] = \begin{cases} 2\frac{\sqrt{y}}{2} & for 0 \leq y \leq 4 \\ 0 & o.w \end{cases}$$

$$\Rightarrow f[\omega(y)] = \begin{cases} \sqrt{y} & for 0 \leq y \leq 4 \\ 0 & o.w \end{cases}$$

$$|J| = \left|\frac{dx}{dy}\right| = \frac{1}{4\sqrt{y}}$$

$$g(y) = \begin{cases} \frac{\sqrt{y}}{4\sqrt{y}} & for\ 0 \le y \le 4 \\ 0 & o.w \end{cases}$$

$$g(y) = \begin{cases} \frac{1}{4} & for\ 0 \le y \le 4 \\ 0 & o.w \end{cases} \qquad g(y) \sim uniform(0,4)$$

Example: If the p.d.f. of x is $f(x) = 2xe^{-x^2}\ 0 < x < \infty$. Determine the p. d. f. of $y = x^2$.

Solution:-

$$f(x) = \begin{cases} 2xe^{-x^2} & for 0 \le x < \infty \\ 0 & o.w \end{cases}$$

$$g(y) = f[\omega(y)].|J|$$

$$y = x^2 \quad \Rightarrow x = \sqrt{y}$$

$$f[\omega(y)] = \begin{cases} 2\sqrt{y}\, e^{-y} & for\ 0 \le y < \infty \\ 0 & o.w \end{cases}$$

$$|J| = \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$$

$$g(y) = \begin{cases} 2\sqrt{y}\, e^{-y} \frac{1}{2\sqrt{y}} & for\ 0 \le y < \infty \\ 0 & o.w \end{cases}$$

$$g(y) = \begin{cases} e^{-y} & for\ 0 \le y < \infty \\ 0 & o.w \end{cases} \qquad g(y) \sim Gamma(1,1)$$

Example: Let $x \sim uniform(0, \alpha)$. Determine the p. d. f. of $Y = cx + d$.

Solution:-

$$f(x) = \begin{cases} \frac{1}{\propto} & for 0 \le x \le \propto \\ 0 & o.w \end{cases}$$

$$g(y) = f[\omega(y)].|J|$$

$$y = [cx + d] \quad \div c$$

$$x = \frac{y - d}{c}$$

$$f[\omega(y)] = \begin{cases} \frac{1}{\propto} & for\ d \leq y \leq c \propto +d \\ 0 & o.w \end{cases}$$

$$|J| = \left|\frac{dx}{dy}\right| = \frac{1}{c}$$

$$g(y) = \begin{cases} \frac{1}{c \propto} & for\ d \leq y \leq c \propto +d \\ 0 & o.w \end{cases}$$

Example: Let $x \sim uniform\ (0,2).$ Find the p.d.f. of $Y = X^2$

Solution:-

$$f(x) = \begin{cases} \frac{1}{2} & for\ 0 \leq x \leq 2 \\ 0 & o.w \end{cases}$$

$$y = x^2 \ \Rightarrow x = \sqrt{y}$$

$$g(y) = f[\omega(y)].|J|$$

$$f[\omega(y)] = \begin{cases} \frac{1}{2} & for\ 0 \leq y \leq 4 \\ 0 & o.w \end{cases}$$

$$|J| = \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$$

$$g(y) = \begin{cases} \frac{1}{2}\ \frac{1}{2\sqrt{y}} & for\ 0 \leq y \leq 4 \\ 0 & o.w \end{cases}$$

$$g(y) = \begin{cases} \frac{1}{4\sqrt{y}} & for\ 0 \leq y \leq 4 \\ 0 & o.w \end{cases}$$