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محاضرات الاحصاء ١

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Central limit theorem

1- Let $X_1 \dots X_n$ be a random sample from a beta distribution with $\alpha = 2$ and $\beta = 3$ find the joint pdf of $Y_1\ and\ Y_n$.

2- Let $X_1 \dots X_n$ a random sample from an exponential population with parameter θ.

Let $Y_1 \dots . Y_n$ be the ordered random variables . Show that the sampling distribution of Y_1 and Y_n are given by

$$f(x)\begin{Bmatrix} \frac{n}{\theta} e_1^{-\frac{ny}{\theta}} & y_1 > 0 \\ 0 & other\ wise \end{Bmatrix}$$

3- Let $X_1 \dots . X_n$ be a random sample with
$f(x) = 3x^3 \quad 0 < x < 1.$
Prove that $U = Y_2/Y_4$ and $V = Y_4$ are indep. .

Limiting Distribution

1- Convergence in Probability

In this section we formalize a way of saying that a sequence of random variables is getting "close" to another random variable.

We will use this concept throughout the lecture.

Definition: let $\{X_n\}$ be a sequence of random variable and let X be a random variable defined on a sample space .

We say that X_n converges in probability to X if for all $\epsilon > 0$

$$\lim_{n\to\infty} P[|X_n - X| \geq \epsilon] = 0$$

Or equivalently ,

$$\lim_{n\to\infty} P[|X_n - X| < \epsilon] = 1$$

If so , we write

$$X_n \xrightarrow{p} X.$$

One way of showing convergence in probability is to use Chebyshev's Theorem

Chebyshev's Theorem:

Let the random variable X have a mean a meat μ and standard deviation σ .

Then for $k > 0$ a constant .

$$P\{|X_n - \mu| < k\sigma\} \geq 1 - \frac{1}{k^2}$$

Example: let $\overline{X_n}$ be a mean of r.s of size n of distribution having this mean " μ" varianse σ^2 then $\overline{X_n} \underset{cs}{\overset{sc}{\rightarrow}} \mu$

solution: $P\{|Y_n - c| < \epsilon\} \geq 1 - \frac{1}{k^2}$ $\quad Y_n = \overline{x_n}\,, c = \mu$

Since $\overline{X_n}$ $\therefore \mu(\overline{X_n}) = \mu, var\ (\overline{x_n}) = \frac{\sigma^2}{n} \rightarrow \sigma_{\overline{X_n}} = \frac{\sigma}{\sqrt{n}}$

$$P\{|X_n - \mu| < \epsilon\} \geq 1 - \frac{1}{k^2} \quad \text{where } \epsilon = k * \frac{\sigma}{\sqrt{n}}$$

$$\left\{\epsilon = k * \frac{\sigma}{\sqrt{n}}\right\} * \frac{\sqrt{n}}{\sigma} \rightarrow k = \frac{\sqrt{n}\epsilon}{\sigma_{\overline{X_n}}}$$

$$P\left\{|X_n - \mu| < \frac{k\ \sigma_{\overline{X_n}}}{\sqrt{n}}\right\} \geq 1 - \frac{1}{(\frac{\sqrt{n}\epsilon}{\sigma_{\overline{X_n}}})^2}$$

$$P\left\{|X_n - \mu| < \frac{k\ \sigma_{\overline{X_n}}}{\sqrt{n}}\right\} \geq 1 - \frac{\sigma^2}{n\epsilon^2}$$

$$\lim_{n\rightarrow\infty}\left[1 - \frac{\sigma^2}{n\epsilon^2}\right] = 1 - \frac{1}{\infty} = 1 - 0 = 1$$

By chebyshev's Theorem

$$\Rightarrow \overline{x_n} \xrightarrow{cs} \mu$$

Example: $Y_n \sim Poi(n)$ show that $\frac{y_n}{n} \overset{c.s}{\rightarrow} 1$

Solution :

$$P\{|Y_n - c| < \epsilon\} \geq 1 - \frac{1}{k^2}$$

$$P\left\{\left|\frac{Y_n}{n} - 1\right| < \epsilon\right\} \geq 1 - \frac{1}{k^2}$$

$$P\{|Y_n - n| < \epsilon\} \geq 1 - \frac{1}{k^2}$$

Since $Y_n \sim Poi(n)$ $\therefore mean\ (Y_n) = n\ \ var(Y_n) = n$

$$\therefore \sigma_{\overline{Y_n}} = \sqrt{n} \quad \rightarrow \quad n\epsilon = k * \sigma_{Y_n} \Rightarrow n\epsilon = k * \sqrt{n}$$

$$k = \frac{n\epsilon}{\sqrt{n}} \Rightarrow k = \sqrt{n}\epsilon$$

$$\therefore P\left\{|Y_n - n| < k\sqrt{n}\right\} \geq 1 - \frac{1}{(\sqrt{n}\epsilon)^2} \quad \rightarrow P\left\{|Y_n - n| < k\sqrt{n}\right\} \geq 1 - \frac{1}{n\epsilon^2}$$

$$\lim_{n\rightarrow\infty} P\left\{|Y_n - n| < k\sqrt{n}\right\} \geq 1 - \frac{1}{n\epsilon^2}$$

$\lim_{n\rightarrow\infty}\left[1 - \frac{1}{n\epsilon^2}\right] = 1 - \frac{1}{\infty} = 1$ by chebyshev's theorem

$$\therefore \frac{Y_n}{n} \overset{cs}{\rightarrow} 1$$

Example: show that $\frac{ns^2}{n-1} \overset{c.s}{\rightarrow} \sigma^2$.

Solution: $\therefore \frac{ns^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$$\epsilon\left(\frac{ns^2}{\sigma^2}\right) = n - 1, var\left(\frac{ns^2}{\sigma^2}\right) = 2(n-1) \Rightarrow S.D = \sqrt{2(n-1)}$$

$$P\{|Y_n - c| < \epsilon\} \geq 1 - \frac{1}{k^2}$$

$$P\left\{\left|\frac{ns^2}{n-1} - c\right| < \epsilon\right\} \geq 1 - \frac{1}{k^2}$$

$$P\left\{\left|\frac{ns^2}{\sigma^2}-(n-1)\right|<\frac{(n-1)\epsilon}{\sigma^2}\right\}\geq 1-\frac{1}{k^2}$$

$$\frac{(n-1)}{\sigma^2}\ \epsilon = k*(S.D) \Longrightarrow \left[\frac{(n-1)\epsilon}{\delta^2}=k*\sqrt{2(n-1)}\right]\div\sqrt{2(n-1)}$$

$$k=\frac{(n-1)\epsilon}{\sigma^2\sqrt{2(n-1)}}\Longrightarrow k=\frac{(n-1)\epsilon}{\sigma^2\sqrt{2}\sqrt{(n-1)}}\Longrightarrow k=\frac{\sqrt{n-1}\epsilon}{\sigma^2\sqrt{2}}$$

$$P\left\{\left|\frac{ns^2}{\sigma^2}-(n-1)\right|<k\sqrt{2(n-1)}\right\}\geq 1-\frac{1}{\left(\frac{\sqrt{n-1}\epsilon}{\sigma^2\sqrt{2}}\right)^2}$$

$$P\left\{\left|\frac{ns^2}{\sigma^2}-(n-1)\right|<k\sqrt{2(n-1)}\right\}\geq 1-\frac{2\sigma^4}{(n-1)\epsilon^2}$$

$$\lim_{n\to\infty}P\left\{\left|\frac{ns^2}{\sigma^2}-(n-1)\right|<k\sqrt{2(n-1)}\right\}\geq 1-\frac{2\sigma^4}{(n-1)\epsilon^2}=1-\frac{1}{\infty}=1$$

$$\therefore \frac{ns^2}{(n-1)}\sim\sigma^2$$

Theorem: $\chi^2_{(n)}\xrightarrow{c.s}c$, then $\frac{\chi^2_{(n)}}{c}\xrightarrow{c.s}1$

Proof: $P\{|Y_n-c|<\epsilon\}\div|c|$

$$\lim_{n\to\infty}\left\{\frac{|Y_n-c|}{|c|}<\frac{\epsilon}{|c|}\right\}$$

$$\lim_{n\to\infty}\left\{\left|\frac{Y_n-c}{c}\right|<\frac{\epsilon}{|c|}\right\}\quad \epsilon'=\frac{\epsilon'}{|c|}$$

$$\lim_{n\to\infty}\left\{\left|\frac{Y_n}{c}-\frac{c}{c}\right|<\epsilon'\right\}=\lim_{n\to\infty}\left\{\left|\frac{Y_n}{c}-1\right|<\epsilon'\right\}$$

$$\therefore \lim_{n\to\infty}\{|Y_n-c|<\epsilon\}=\left\{\left|\frac{Y_n}{c}-1\right|<\epsilon'\right\}$$

Since $\chi_n\xrightarrow{c.s}c\Rightarrow\lim_{n\to\infty}\{|Y_n-c|<\epsilon\}=1\to\lim_{n\to\infty}\left\{\left|\frac{Y_n}{c}-1\right|<\epsilon'\right\}=1$