



**Republic of Iraq Ministry of Higher
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محاضرات الاحصاء ١

مدرس المادة : الاستاذ المساعد الدكتور

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The Distribution of \bar{x} : Let x_1, x_2, \dots, x_n are independent and identically distributed normal random variables with mean μ and variance σ^2 , then the way to find the Distribution of \bar{X} is

$$m_x(t) = \text{EXP} \left\{ \mu t + \frac{1}{2} \sigma^2 r^2 \right\}$$

$$x_{\bar{x}}(t) = E(e^{t\bar{x}}) = E\left(e^{\frac{t}{n}(x_1+x_2+\dots+x_n)}\right) = E\left(e^{\frac{t}{n}x_1} + \frac{t}{n}x_2 + \dots + \frac{t}{n}x_n\right)$$

$$= E\left(e^{\frac{t}{n}x_1}\right)E\left(e^{\frac{t}{n}x_2}\right)\dots E\left(e^{\frac{t}{n}x_n}\right)$$

$$= m_{x_1}\left(\frac{t}{n}\right) m_{x_2} \dots m_{x_n}\left(\frac{t}{n}\right)$$

$$= \text{EXP}\left\{ \mu \frac{t}{n} + \frac{1}{2n^2} \sigma^2 r^2 \right\} \cdot \text{EXP}\left\{ \mu \frac{t}{n} + \frac{1}{2n^2} \sigma^2 r^2 \right\} \dots \text{EXP}\left\{ \mu \frac{t}{n} + \frac{1}{2n^2} \sigma^2 r^2 \right\}$$

$$= \text{EXP}\left\{ n \mu \frac{t}{n} + \frac{1}{2n^2} \sigma^2 r^2 \right\} = \text{EXP}\left\{ \mu t + \frac{1}{2n} \sigma^2 r^2 \right\}$$

That is a moment generating function of $N\left(\mu, \frac{\sigma^2}{n}\right)$, Thus

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

That is

$$f(\bar{X}) = \sqrt{\frac{n}{2n}} \frac{1}{\sigma} e^{\frac{n(\bar{x}-\mu)^2}{2\sigma^2}} \quad ; -\infty < \bar{X} < \infty$$

To drive the mean and var . of \bar{X} :

$$E(\bar{x}) = E\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \frac{1}{n} (E(x_1) + E(x_2) + \dots + E(x_N))$$

$$= \frac{1}{n} (\mu + \mu + \dots + \mu) = \frac{1}{n} n\mu = \mu$$

$$\therefore E(\bar{X}) = \mu$$

$$\text{Var}(\bar{x}) = \left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \frac{1}{n^2} (\text{var}(x_1) + \text{var}(x_2) + \dots + \text{var}(x_n))$$

$$= \frac{1}{n} (\sigma^2 + \sigma^2 \dots \dots + \sigma^2)$$

$$= \frac{1}{n^2} n \sigma^2 = \frac{1}{n} \sigma^2$$

$$\therefore \text{var } (\bar{x}) = \frac{1}{n} \sigma^2$$

Example: Let x_1, x_2, \dots, x_n are independent and identically distributed $G(\alpha, \beta)$, Find The Distribution of \bar{X} ?

Solution:

$$m_x(t) = (1 - \beta t)^{-\alpha}$$

$$m_{\bar{x}}(t) = E(e^{t\bar{x}}) = E\left(e^{\frac{t}{n}(x_1+x_2+\dots+x_n)}\right) = E\left(e^{\frac{t}{n}x_1 + \frac{t}{n}x_2 + \dots + \frac{t}{n}x_n}\right)$$

$$= E\left(e^{\frac{t}{n}x_1}\right)E\left(e^{\frac{t}{n}x_2}\right)\dots\dots E\left(e^{\frac{t}{n}x_n}\right)$$

$$= m_{x_1}\left(\frac{t}{n}\right) m_{x_2}\left(\frac{t}{n}\right) \dots \dots m_{x_n}\left(\frac{t}{n}\right)$$

$$= \left(1 - \beta \frac{t}{n}\right)^{-\alpha} \left(1 - \beta \frac{t}{n}\right)^{-\alpha} \dots \dots \left(1 - \beta \frac{t}{n}\right)^{-\alpha} \rightarrow \left(1 - \frac{\beta}{n}t\right)^{-n\alpha}$$

That is a moment generating function of $G(n\alpha, \frac{\beta}{n})$. Thus

$$\bar{X} \sim G\left(n\alpha, \frac{\beta}{n}\right)$$

That is

$$f(\bar{x}) = \frac{n^\alpha}{\beta^\alpha \Gamma(n\alpha)} (\bar{x})^{n\alpha-1} e^{\frac{n\bar{x}}{\beta}} ; 0 < \bar{x} < \infty$$

To drive the mean and var . of \bar{X} :

$$E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \frac{1}{n} (E(x_1) + E(x_2) + \dots + E(x_n))$$

$$= \frac{1}{n} (\alpha\beta + \alpha\beta + \dots + \alpha\beta) = \frac{1}{n} n\alpha\beta = \alpha\beta$$

$$\therefore E(\bar{X}) = \alpha\beta$$

$$\begin{aligned} \text{var}(\bar{X}) &= \text{var}\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \frac{1}{n^2} (\text{var}(x_1) + \text{var}(x_2) + \dots + \\ &\quad \text{var}(x_n)) = \frac{1}{n} (\alpha\beta^2 + \alpha\beta^2 + \dots + \alpha\beta^2) \\ &= \frac{1}{n^2} n\alpha\beta^2 = \frac{1}{n} \alpha\beta^2 \end{aligned}$$

$$\therefore (\bar{X}) = \frac{1}{n} \alpha\beta^2$$

Example : Let x_1, \dots, x_2 , be a random variable of size 25 from $\sim (75, 100)$ compute, $p(71 < \bar{X} < 79)$?

Solution :

$$N=25$$

Hint $F(2)=2.0$

$$\bar{X} = \left(\mu, \frac{s^2}{n} \right)$$

$$\bar{X} = \left(75, \frac{100}{25} \right)$$

$$\bar{X} = (75, 4)$$

$$\mu = 75 \quad s^2 = 4 \longrightarrow s = 2$$

$$p\left(\frac{x_1-\mu}{s} < \frac{\bar{x}-\mu}{s} < \frac{x_2-\mu}{s}\right)$$

$$p\left(\frac{71-75}{2} < \frac{\bar{X}-\mu}{s} < \frac{79-75}{2}\right).$$