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# محاضرات الاحصاء ١

مدرس المادة : الاستاذ المساعد الدكتور

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## The Distribution of $S^2$

Theorem:- Let $X_1 . X_2 \dots . X_n$ be observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$ Then the sample mean .

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

and the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

are independent and

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sigma^2}$$

Proof :- we are not prepared to prove the independence of $\bar{X}$ and $S^2$ at this time, so we accept it without proof here . To prove the second part . note that

$$w = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n} \left[\frac{(X_i - \bar{X}) + (\bar{X} - \mu)}{\sigma}\right]^2$$

$$= \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 + \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

because the cross-product term is equal to

$$2 \sum_{1}^{n} \frac{(\bar{X} - \mu)(X_i - \bar{X})}{\sigma^2} = \frac{2(\bar{X} - \mu)}{\sigma^2} \sum_{1}^{n} (X_i - \bar{X}) = 0$$

But $Y_i = \frac{(\bar{X} - \mu)}{\sigma^2}, i = 1.2.3 \dots . n$

are standardized normal variables that are independent. Hence $w = \sum_{1}^{n} Y_i^2$ is $\chi^2(n)$ by corollary 5.4-3 Moreover

since $\bar{X}$ is $N(\mu, \frac{\sigma^2}{n})$ it follows that

$$Z^2 = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 = \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

is $\chi^2(1)$ by Theorem 3.3-2 In this notation. Equation 5.5-1becomes

$$w = \frac{(n-1)s^2}{\sigma^2} + z^2$$

However from the face that $\bar{X}$ and $S^2$ are independent it follows that$Z^2$ and $S^2$ are also independent In the mgf of W this independence permits us to write

$$E[e^{tw}] = E\left[e^{t(\frac{(n-1)s^2}{\sigma^2} + z^2)}\right] = E\left[e^{t(\frac{(n-1)s^2}{\sigma^2})} e^{tZ^2}\right] = E\left[e^{t(\frac{(n-1)s^2}{\sigma^2})}\right] E\left[e^{tZ^2}\right] .$$

Since W and $z^2$ have chi-square distribution we can substitute their mgfs to obtain $(1-2t)^{-n/2} = E\left[e^{t(\frac{(n-1)s^2}{\sigma^2})}\right](1-2t)^{-1/2}$

Equivalently we have $E\left[e^{t(\frac{(n-1)s^2}{\sigma^2})}\right] = (1-2t)^{-(n-1)/2} \quad t<1/2$

This of course is the mgf of a$\chi^2(n-1)$ variable accordingly($\frac{(n-1)s^2}{\sigma^2}$ has that distribution

**<u>Example</u>**:- If $\bar{X}\sim N\left(\mu,\frac{\sigma^2}{n}\right)$ Show that $Z=\left[\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right]\sim N(0,1)$

**<u>Solution:</u>**

Since $\bar{X}\sim N\left(\mu,\frac{\sigma^2}{n}\right)$

$$f(\bar{X}) = \frac{1}{\frac{\sigma^2}{n}\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{(\bar{X}-\mu)^2}{\frac{\sigma^2}{n}}\right)} \quad -\infty<\bar{X}<\infty$$

$$f(\bar{X})=\frac{\sqrt{n}}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{(\bar{X}-\mu)^2}{\frac{\sigma}{\sqrt{n}}}\right)} -\infty<\bar{X}<\infty$$

$$Z=\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

$$M_z(t) = E(e^{tz})$$

$$=E(e^{t\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)})$$

$$E\left(e^{t\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)}\right)=\int_{-\infty}^{\infty} e^{t\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)}.\frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\ e^{\frac{-1}{2}\left(\frac{(\bar{X}-\mu)^2}{\frac{\sigma^2}{n}}\right)}d\bar{X}$$

$$\text{let}\left[y=\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right]\rightarrow\frac{\sigma y}{\sqrt{n}}=\bar{X}-\mu$$

$$\bar{X}=\frac{\sigma y}{\sqrt{n}}+\mu\ \rightarrow d\bar{X}=\frac{\sigma}{\sqrt{n}}dy$$

$$E(e^{tz}) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)}.\ e^{\frac{-1}{2}\left(\frac{(\bar{X}-\mu)^2}{\frac{\sigma^2}{n}}\right)}d\bar{X}$$

$$E(e^{tz}) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ty}\, e^{\frac{-1}{2}y}\frac{\sigma}{\sqrt{n}}dy$$

$$E(e^{tz}) = \frac{\sigma}{\sqrt{n}}\frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\frac{y^2-2ty}{2})}dy$$

$$E(e^{tz}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\frac{y^2-2t+t^2-t^2}{2})}\ dy$$

$$E(e^{tz}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{y^2-2t+t^2}{2}\right)-\frac{t^2}{2}}\,dy$$

$$E(e^{tz}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-t)^2}{2}} e^{\frac{t^2}{2}} dy$$

$$E(e^{tz}) = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\frac{(y-t)^2}{2})} dy$$

Let h=y-t $\rightarrow dh = dy$

$$E(e^{tz}) = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}h^2}\,dh$$

$$E(e^{tz}) = e^{\frac{t^2}{2}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}h^2}\,dh \rightarrow \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}h^2} = 1 \sim N(0,1)$$

$E(e^{tz})= e^{\frac{t^2}{2}} \sim N(0,1)$

$Z=\left[\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right] \sim N(0,1)$

## Student t-distribution:-

Theorem :-Let $T = \frac{Z}{\sqrt{\frac{U}{r}}}$

where Z is a random variable that is $N(0,1)$, U is a random variable that is $X^2(r)$ and Z and U are independent . Then T has a t distribution with pdf

$$f(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \quad -\infty < t < \infty$$

proof :- The joint pdf of Z and U is

$$g(z,u)=\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\frac{r}{2}}} u^{\frac{r}{(2-1)}} e^{-\frac{u}{2}}$$

the cdf $F(t)=P(T\leq t)$ of Tis given by

$$F(t)=P\left(\frac{Z}{\sqrt{\frac{U}{r}}} \leq t\right)$$

$$= P\left(Z \leq \sqrt{\frac{U}{rt}}\right)$$

$$=\int_0^{\infty}\int_{-\infty}^{\sqrt{\left(\frac{u}{r}\right)}t} g(z,u)\,dz\,du. \text{That is } F(t)= \frac{1}{\sqrt{\pi}\Gamma\left(\frac{r}{2}\right)}\int_0^{\infty}\left[\int_{-\infty}^{\sqrt{\left(\frac{u}{r}\right)}t} \frac{e^{-\frac{z^2}{2}}}{\frac{2(r+1)}{2}}\,dz\right] u^{\frac{r}{2}} e^{-\frac{u}{2}}\,du$$

the pdf of T is the derivative of the cdf , so, applying the fundamental theorem of calculus to the inner integral

we find that $f(t) = F\acute{}(t) = \frac{1}{\sqrt{\pi}\Gamma\left(\frac{r}{2}\right)} \int_0^{\infty} \frac{e^{-(\frac{u}{2})(\frac{t^2}{r})}}{2^{(r+1)/2}} \sqrt{\frac{u}{r}}\, u^{\frac{r}{2}-1} e^{-u/2} du$

$=\frac{1}{\sqrt{\pi}\Gamma\left(\frac{r}{2}\right)} \int_0^{\infty} \frac{u^{\frac{r+1}{2}-1}}{2^{(r+1)/2}} e^{-(\frac{u}{2})(\frac{1+t^2}{r})} du$

In the integral, make the change of variables y=(1+$t^2/r$)u , so that

$\frac{du}{dy} = \frac{1}{1+t^2/r}$

Thus, $f(t) = \frac{\Gamma[(r+1)]/2}{\sqrt{\pi r}\Gamma(\frac{r}{2})} \left[\frac{1}{(1+t^2/r)^{(r+1)/2}}\right] \int_0^{\infty} \frac{y^{(r+1)/(2-1)}}{\Gamma[\frac{r+1}{2}]2^{(r+1)/2}} e^{-y/2} dy$

The integral in this last expression for f(t) is equal to 1 because the integrand is like the pdf of a chi-square distribution with r+1 degrees of freedom . Hence , the pdf is

$f(t) = \frac{\Gamma[{}^{(r+1)}/_{2}]}{\sqrt{\pi r}\Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \quad -\infty < t < \infty$

**Example:**if T ~t(10) then what is the probability that T is at least 2.228?

**Solution:**

$P(T \geq 2 \cdot 228) = 1 - P(T < 2 \cdot 228)$

$= 1 - 0 \cdot 975$ (from t- table)

$= 0 \cdot 025$