Republic of Iraq Ministry of Higher Education & Research

University of Anbar

College of Education for Pure Sciences

Department of Mathematics

محاضرات الاحصاء ١

مدرس المادة : الاستاذ المساعد

الدكتور فراس شاكر محمود

Limiting Moment-Generating Functions

To find the limiting distribution function of a random variable V by use of the definition of limiting distribution function obviously requires that we know F,(y) for each positive integer n. This is precisely the problem we should like to avoid. If it exists, the moment-generating function that corresponds to the distribution function F,(y) often provides a convenient method of determining the limiting distribution function. To emphasize that the distribution of a random variable Y, depends upon the positive integer n, in this lecture we shall write the moment-generating function of Y, in the form M(t; n). The following theorem, which is essentially Curtiss' modification of a theorem of Lévy and Cramér, explains how the moment-generating function may be used in problems of limiting distributions. A proof of the theorem requires a knowledge of that same facet of analysis that permitted us to assert that a moment-generating function, when it exists, uniquely determines a distribution. Accordingly, no proof of the theorem will be given.

Theorem 1. Let the random variable Y_n , have the distribution function $F_n(y)$ and the moment-generating function M(t ; n) that exists for $-h < t < h$ for all n. If there exists a distribution function F(y), with corresponding moment-generating function M(t), defined for $|t| \leq h_1 < h$, such that $\lim_{n\to\infty} M(t;n) = M(t)$, then Y_n , has a limiting distribution with distribution function F(y). Several illustrations of the use of **Theorem 1.** In some of these examples it is convenient to use a certain limit that is established in some courses in advanced calculus. We refer to a limit of the form

$$\lim_{n\to\infty}\left(1+\frac{b}{n}+\frac{\varphi(n)}{n}\right)^{cn}$$

where b and c do not depend upon n and where $\lim_{n\to\infty}(\varphi(n)) = 0$. then

$\lim_{n\to\infty}\left(1+\frac{b}{n}+\frac{\varphi(n)}{n}\right)^{cn} = \lim_{n\to\infty}\left(1+\frac{b}{n}\right)^{cn} = e^{bc}$.

Example:

$\lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3}{n^{3/2}}\right)^{-n/2} = \lim_{n\to\infty}\left(1-\frac{t^2}{n}+\frac{t^3/\sqrt{n}}{n^{3/2}}\right)^{-n/2}$

Here b= - t^2 ، c= $\frac{-1}{2}$ ، and $\varphi(n) = t^3/\sqrt{n}$

Accordingly for every fixed value of t, the limit is $e^{t^2/2}$

Theorem 2. let $Y_n \sim b(n،p)$ show that the limit of Y_n as $n \to \infty$.

Proof:

Since $Y_n \sim b(n،p)$

So $M_{Y_n}\left(t،n\right) = (pe^t + q)^n$　　　　q=1-p

$$\mu = np \to p = \frac{\mu}{n}$$

$M_{Y_n}\left(t،n\right) = (pe^t + 1 - p)^n$　　　　سحب p عامل مشترك

$M_{Y_n}\left(t،n\right) = (p(e^t - 1) + 1)^n$　　　　نعوض قيمة p

$$M_{Y_n}\left(t،n\right) = \left(\frac{\mu}{n}(e^t - 1) + 1\right)^n$$

$$\left(1+\frac{x}{n}\right)^n = e^x \text{ where } x = \mu(e^t - 1)$$

$$\therefore M_{Y_n}(t,n) = \left(\frac{\mu(e^t-1)}{n}+1\right)^n = e^{\mu(e^t-1)}$$

$$Y_n = e^{\mu(e^t-1)} \qquad \sim pois(\mu)$$

$$\therefore \lim_{n\to\infty} Y_n \sim pois(\mu)$$

Example: let $Z_n \sim p(n)$ find the limiting distribution of $Y_n = \frac{Z_n - n}{\sqrt{n}}$?

Solution :

$$M_{Y_n}(t,n) = E(e^{Y_n}) = E\left(e^{t\frac{Z_n-n}{\sqrt{n}}}\right)$$

$$= e^{t\frac{-n}{\sqrt{n}}} \quad E\left(e^{t\frac{Z_n}{\sqrt{n}}}\right)$$

Since $Z_n \sim p(n) \rightarrow M_{Z_n} = e^{n(e^t-1)}$

$$= e^{t\frac{-n}{\sqrt{n}}} \quad e^{n(e^t-1)} = e^{-t\sqrt{n}} \; e^{ne^{\frac{t}{\sqrt{n}}}-n}$$

$$M_{Y_n} = e^{-\sqrt{n}\ t-n+ne^{\frac{t}{\sqrt{n}}}}$$

$$e^{\frac{t}{\sqrt{n}}} = 1 + \frac{t}{\sqrt{n}} + \frac{1}{2!}\left(\frac{t}{\sqrt{n}}\right)^2 + \cdots$$

$$M_{Y_n} = e^{-\sqrt{n}\ t-n+n\left(1+\frac{t}{\sqrt{n}}+\frac{1}{2!}\left(\frac{t}{\sqrt{n}}\right)^2+\cdots\right)}$$

$$M_{Y_n} = e^{-\sqrt{n}\ t-n+n+\sqrt{n}\ t+\frac{t^2}{2}+\cdots}$$

$$M_{Y_n} = e^{\frac{t^2}{2}+\frac{\emptyset(n)}{n}} \qquad \emptyset(n) \to 0$$

$$M_{Y_n} = e^{\frac{t^2}{2}} \quad \sim \; N\left(0،1\right)$$

Example: Let Y_n denote the n^{th} order statistic of r.s from a distribution of the continuous type that has distribution function $F(x)$ and p.d.f. $f(x)$ find the limiting distribution of $Z_n = n[1 - F(Y_n)]$?

Solution :

Since Y_n $\therefore$ order largest

$$g(Y_n) = n[F(Y_n)]^{n-1}\, f(Y_n)$$

Since $Z_n = n[1 - F(Y_n)]$

$$Z_n = n - nF(Y_n)$$

$$nF(Y_n) = n - Z_n \to \; F(Y_n) = 1 - \frac{Z_n}{n}$$

هذه الدالة التوزيعية

يجب ان نجد الدالة الاحتمالية اي اشتقاق الدالة التوزيعية بالنسبة ل Z_n

$$F(Y_n)\frac{dY_n}{dZ_n} = -\frac{1}{n}$$

$$\frac{dY_n}{dZ_n} = \frac{-1}{nf(Y_n)} \to \left|\frac{dY_n}{dZ_n}\right| = \left|\frac{-1}{nf(Y_n)}\right| = \frac{1}{nf(Y_n)}$$

لان

$$h(Z_n) = g(Y_n)|J|$$

$$h(Z_n) = n\left[1-\frac{Z_n}{n}\right]^{n-1} f(Y_n)\frac{1}{nf(Y_n)} \rightarrow h(Z_n) = \left[1-\frac{Z_n}{n}\right]^{n-1}$$

$$\lim_{n\to\infty}\left[1-\frac{Z_n}{n}\right]^{n-1} \rightarrow \lim_{n\to\infty}\left[1-\frac{Z_n}{n}\right]^{n} \lim_{n\to\infty}\left[1-\frac{Z_n}{n}\right]^{-1}$$

$$(1-0)\lim_{n\to\infty}\left[1-\frac{Z_n}{n}\right]^{n} = e^{-Z_n} \sim G\left(1،1\right) \text{ or Ex }(1)$$