Republic of Iraq Ministry of Higher Education & Research

University of Anbar

College of Education for Pure Sciences

Department of Mathematics

محاضرات الاحصاء ١

مدرس المادة : الاستاذ المساعد الدكتور

فراس شاكر محمود

1- Moment Method:

Let $X_1 X_2, \dots, X_n$ be a random sample from a population X with probability density function $f(x; \theta_1, \theta_2, \dots, \theta_m)$, where $\theta_1, \theta_2, \dots, \theta_m$ are m unknown parameters. Let

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x; \theta_1, \theta_2, \dots, \theta_m) dx$$

Be the k^{th} population moment about 0. Further, let

$$M_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

Be the k^{th} sample moment about 0.

In moment method, we find the estimator for the parameters $\theta_1, \theta_2, \dots, \theta_m$ by equating the first m population moments (if they exist) to the first m sample moments, that is

$$E(X) = M_1$$

$$E(X^2) = M_2$$

$$E(X^3) = M_3$$

$$\vdots$$

$$E(X^m) = M_m$$

The moment method is one of the classical methods for estimating pa-rameters and motivation comes from the fact that the sample moments are in some sense estimates for the population moments. The moment method was first discovered by British statistician Karl Pearson in 1902. Now we provide some examples to illustrate this method.

<u>Example</u>. Let $X \sim N(\mu, \sigma^2)$ and $X_1, X_2, \dots, X_n$ be a random sample of size n from the population X. What are the estimators of the population parameters λ and σ^2 if we use the moment method?

Solution: Since the population is normal, that is

$$X \sim N(\mu, \sigma^2)$$

We know that

$$E(X) = \mu$$

$$E(X^2) = \sigma^2 + \mu^2$$

Hence

$$\mu = E(X)$$

$$= M_1$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$= \bar{X}.$$

Therefore, the estimator of the parameter μ is $\bar{X}$, that is

$$\hat{\mu} = \bar{X}.$$

Next, we find the estimator of σ^2 equating $E(X^2)$ to M_2. Note that

$$\sigma^2 = \sigma^2 + \mu^2 - \mu^2$$

$$= E(X^2) - \mu^2$$

$$= M_2 - \mu^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2 \,.$$

The last line follows from the fact that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i^2 - 2X_i\,\bar{X} + \bar{X}^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i=1}^{n} 2X_i\,\bar{X} + \frac{1}{n}\sum_{i=1}^{n} \bar{X}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\bar{X}\frac{1}{n}\sum_{i=1}^{n} X_i + \bar{X}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\bar{X}\bar{X} + \bar{X}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

Thus, the estimator of σ^2 is $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, that is

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

<u>Example</u>. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a population X whit probability density function

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & if 0 < x < 1 \\ 0 & otherwise, \end{cases}$$

Where 0 < θ < ∞ is an unknown parameter. Using the method of moment find an estimator of θ ? If $x_1 = 0.2, x_2 = 0.6, x_3 = 0.5, x_4 = 0.3$ is a random sample of size 4, then what is the estimate of θ ?

<u>Solution</u> To find an estimator, we shall equate the population moment to the sample moment. The population moment E(X) is given by

$$E(X) = \int_0^1 x f(x;\theta)dx$$

$$= \int_0^1 x\theta x^{\theta-1}dx$$

$$= \theta \int_0^1 x^\theta \, dx$$

$$= \frac{\theta}{\theta + 1} \left[x^{\theta+1}\right]_0^1$$

$$= \frac{\theta}{\theta + 1}$$

We know that $M_1 = \bar{X}$. now setting M_1 equal to E(X) and solving for θ, we get

$$\bar{X} = \frac{\theta}{\theta + 1}$$

That is

$$\theta = \frac{\bar{X}}{1 - \bar{X}}$$

Where $\bar{X}$ is the sample mean. Thus, the statistic $\frac{\bar{X}}{1-\bar{X}}$ is an estimator of the parameter θ . Hence

$$\hat{\theta} = \frac{\bar{X}}{1 - \bar{X}}$$

Since $x_1 = 0.2, x_2 = 0.6, x_3 = 0.5, x_4 = 0.3$, we have $\bar{X} = 0.4$ and

$$\hat{\theta} = \frac{0.4}{1 - 0.4} = \frac{2}{3}$$

Is an estimate of the θ .

<u>Example</u>. Let $X \sim poisson(\lambda)$ find Moment Estimate of λ ?

<u>Solution:</u> Since $X \sim poisson(\lambda)$

$$f(X) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{X!} & for\ x = 0,1, \dots \\ 0 & otherwise \end{cases}$$

$$E(x) = \lambda, Var(x) = \lambda, \bar{X} = \frac{\sum x_i}{n}$$

$$E(X) = \bar{X}$$

$$\lambda = \frac{\sum x_i}{n}$$

$$\hat{\lambda} = \bar{X}$$

Example: Let X~Binomaill$(20, \mathrm{p})$ find Moment Estimate of p?

Solution: Since X~Binomaill$(20, \mathrm{p})$

$$f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & for\ x = 0,1,\dots,n \\ 0 & 0 \le p \le 1 \end{cases}$$

$$E(x) = np\ , Var(x) = npq$$

$$E(x) = 20p\ , \bar{X} = \frac{\sum x_i}{n}$$

$$E(x) = \bar{X} \rightarrow \frac{\left[20p = \frac{\sum x_i}{n}\right]}{20}$$

$$p = \frac{1}{20} \cdot \frac{\sum x_i}{n} \rightarrow p = \frac{\bar{X}}{20}$$

$$\hat{p} = \frac{1}{20}\bar{X}$$

Example: Let $X_1, X_2, \dots, X_n$ a random variable sample from Uniform (0,θ) find the Moment Estimate of θ ?

Solution: Since U(0,θ)

$$f(x) = \begin{cases} \frac{1}{b-a} & for\ a \le x \le b \\ 0 & otherwise \end{cases}$$

$$E(x) = \frac{a+b}{2}$$

$$E(x) = \frac{0+\theta}{2}$$

$$E(x) = \bar{X}$$

$$\frac{\theta}{2} = \bar{X} \rightarrow \theta = 2\bar{X}$$

$$\hat{\theta} = 2\bar{X}$$

$$E(X^2) = \int X^2 \cdot f(x)dx \rightarrow \int_0^{\theta} X^2 \cdot \frac{1}{\theta} dx$$

$$E(X^2) = \left[\frac{1}{\theta} \cdot \frac{X^3}{3}\right]_0^{\theta} \rightarrow E(X^2) = \frac{\theta^3}{3\theta}$$

$$E(X^2) = \frac{\theta^2}{3}$$

$$E(X^2) = \overline{X^2}$$

$$\frac{\theta^2}{3} = \frac{\sum X_i^2}{n} \rightarrow n\theta^2 = 3\sum X_i^2 \rightarrow \theta^2 = \frac{3\sum X_i^2}{n}$$

$$\hat{\theta} = \sqrt{\frac{3\sum X_i^2}{n}}$$

<u>Example:</u> Let $X_1, X_2, \ldots, X_n$ be a random variable sample of size n from distribution, with p.d.f

$$f(x, \alpha, \theta) = \begin{cases} \frac{\theta}{\theta^{\alpha}} X^{\alpha} & for\ 0 < x < \theta \\ 0 & otherwise \end{cases}$$

$$\alpha > 0, \theta > 0$$

Suppose α is known find the moment estimator of θ , $\hat{\theta}$ and unbiased estimator of θ ?

<u>Solution:</u>

$$E(X) = \int x \cdot f(x)dx \rightarrow E(X) = \int_0^{\theta} \frac{\alpha}{\theta^{\alpha}} x^{\alpha-1} \cdot x dx$$

$$E(X) = \int_0^{\theta} \frac{\alpha}{\theta^{\alpha}} \cdot x \cdot x^{-1} \cdot x^{\alpha} dx \; \to E(X) = \int_0^{\theta} \frac{\alpha}{\theta} \cdot x^{\alpha} dx$$

$$E(X) = \left[\frac{\alpha}{\theta^{\alpha}} \cdot \frac{x^{\alpha+1}}{\alpha+1}\right]_0^{\theta} \to E(X) = \frac{\alpha(\theta)^{\alpha+1}}{\theta^{\alpha} \cdot (\alpha+1)} = \frac{\alpha\theta^{\alpha}\theta}{\theta^{\alpha}(\alpha+1)} = \frac{\alpha\theta}{\alpha+1}$$

$$E(X) = \bar{X}$$

$$\frac{\alpha\theta}{\alpha+1} = \bar{X} \;\; \to \frac{[\alpha\theta = (\alpha+1)\bar{X}]}{\alpha}$$

$$\hat{\theta} = \frac{(\alpha+1)\bar{X}}{\alpha} \qquad \to \qquad , \bar{X} = \frac{\alpha\theta}{\alpha+1}$$

$$E(\hat{\theta}) = E\left[\frac{\alpha+1}{\alpha}\bar{X}\right]$$

$$\to \frac{\alpha+1}{\alpha} E(\bar{X})$$

$$\frac{\alpha+1}{\alpha}\left[\frac{\alpha\theta}{\alpha+1}\right]$$

$$E(\hat{\theta}) = \theta$$

$\hat{\theta}$ is not unbiased estimator of θ .

2- Maximum Likelihood Estimator:

Let $L(\theta) = L(\theta; x_1,\ldots,x_n)$

Be the likelihood function for the random variables $X_1, X_2, \ldots, X_n$. if θ (where $\theta = \vartheta\,(x_1, x_2, \ldots,x_n)$ is a function of the observations $x_1,\ldots,x_n$) is the Value of θ in Θ which maximum L(θ). Then $\Theta = \vartheta(X_1,X_2,\ldots,X_n)$ is the Maximum likelihood estimator of $\theta = \vartheta\,(x_1,\ldots,x_n)$ is the maximum likelihood estimate of θ for the example $x_1,\ldots,x_n$.

The most likelihood important cases which we shall consider are those in which $X_1,X_2,\ldots,X_n$ is a random sample from some density $f(x;\theta)$, so that the likelihood function is

$L(\theta) = f(x_1;\theta)\, f(x_2;\theta) \ldots f(x_n;\theta)$.

Many likelihood functions satisfy regularity conditions; so the maximum-likelihood estimator in the solution of the equation.

$$\frac{dL(\theta)}{d\theta} = 0$$

Also, L(θ) and log L(θ) have their maxima at the same value of θ, and it is sometimes easier to find the maximum of the logarithm of the likelihood, if the likelihood function contains (k) parameters, that is

If $L(\theta_1,\theta_2,\ldots,\theta_k) = \prod_{f=1}^{n} f(x_i; \theta_1, \theta_2, \ldots, \theta_k)$

Then the maximum-likelihood estimators of the parameters $\theta_1,\theta_2,\ldots, \theta_k$ are the random variables $\Theta_1 = \vartheta_1(X_1,\ldots,X_n)$, … , $\Theta_٢ = \vartheta_2(X_1,\ldots,X_n)$, …. $\Theta_k = \vartheta_k (X_1,\ldots,X_n)$, where $\theta_{1,}\theta_2,\ldots,\theta_k$ are the values in Θ which maximize $L(\theta_{1,}\theta_2,\ldots,\theta_k)$.

If certain regularity conditions are satisfied, the point where the likelihood is a maximum is a solution of the (k) equation

$$\frac{\vartheta L(\theta_1, \ldots, \theta_k)}{\vartheta\theta_1} = 0$$

$$\frac{\vartheta L(\theta_2, \ldots, \theta_k)}{\vartheta\theta_2} = 0$$

$$\frac{\vartheta L(\theta_1, \ldots, \theta_k)}{\vartheta\theta_k} = 0$$

In this case it may also be easier to work with the logarithm of the likelihood,

We shall illustrate these definitions with some examples.

Example 1. Let $x_1, x_2,\ldots, x_n$ a random variable sample ~ Geometric (p) find Maximum likelihood estimator of (p)

<u>Solution:</u>

Since $x_1,\ldots, x_n$ ~ G(p)

$$f(x) = \begin{cases} P(1-P)^{x-1} & \text{For } x=1,2,\ldots \quad 0 \le P \le 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$f(x_1, \dots, x_n, P) = f(x_1, P) \,.\, f(x_2, P) \dots \dots . f(x_n, P)$$

$$f((x_1, \dots, x_n, P) = P(P-1)^{x-1} \,.\, P(P-1)^{x_2-1} , \dots . , f(x_n, P)$$

$$g(x_1, \dots, x_n, P) = \; P^n \, (1-P)^{\sum x_i - n}$$

$$g(x_n, P) = P^n (1-P)^{\sum x_i - n}$$

$$\ln g(x_n, P) = \ln[P^n] + \ln(1-P)^{\sum x_i - n}$$

$$\ln g\,(x_n, P) = n \ln(P) + (\, \sum x_i - n) \ln\,(1-P)$$

$$\frac{d \ln g(x_n, P)}{dP} = n. \frac{1}{P} + (\sum x_i - n). \frac{-1}{1-P}$$

$$\left[\frac{n}{P} - \left(\sum_{i=1}^{n} x_i - n\right) \frac{1}{1-P}\right] * \; \frac{1-P}{n}$$

$$\frac{1-P}{P} = \frac{\sum x_i - n}{n}$$

$$\frac{1}{P} - \; \frac{P}{P} = \; \frac{\sum_{i=1}^{n} x_i}{n} - \; \frac{n}{n} ==> \; \frac{1}{P} - 1 = \frac{\sum x_i}{n} - 1$$

$$\frac{1}{P} = \; \frac{\sum x_i}{n} ==> P \sum_{i=1}^{n} x_i = n$$

$$P = \frac{n}{\sum_{i=1}^{n} x_i} ==> P = \frac{1}{\bar{x}}$$

$$\hat{P} = \frac{1}{\bar{x}}$$

Example Let $x_1, \dots, x_n$ ~ Poisson (λ) find Maximum-likelihood estimator of λ?

Solution: Since $x_1, \dots, x_n$ ~ Poisson (λ)

$$f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x_i} & \text{For x=0,1,...} \\ 0 & \text{Otherwise} \end{cases}$$

$$f(x_1, \dots, x_n, \lambda) = f(x_1, \lambda) \,.\, f((x_2, \lambda) \dots \dots f((x_n, \lambda)$$

$$f(x_1, \dots, x_n, \lambda) = \frac{\lambda^{x_1} e^{-\lambda}}{x_1!} \,.\, \frac{\lambda^{x_2} e^{-\lambda}}{x_2!} \dots \dots .. \frac{\lambda^{x_n} e^{-\lambda}}{x_n!}$$

$$g(x_n, \lambda) = \frac{e^{-n\lambda} \lambda^{\sum_{1=1}^{n} x_i}}{\sum_{i=1}^{n} x_i\, !}$$

$$\ln g(x_n, \lambda) = \ln \left[\frac{e^{-n\lambda} \lambda^{\sum x_i}}{\sum x_i!} \right]$$

$$\ln g\,(x_n, \lambda) = \ln e^{-n\lambda} + \ln \lambda^{\sum x_i} - \ln \sum_{i=1}^{n} x_i\, !$$

$$\ln g\,(x_n, \lambda) = -n\lambda + \sum_{i=1}^{n} x_i \ln \lambda - 0$$

$$\ln g(x_n, \lambda) = -n\lambda + \sum_{i=1}^{n} x_i \ln \lambda$$

$$\frac{d \ln g(x_n, \lambda)}{\lambda} = -n + \sum_{i=1}^{n} x_i \,.\, \frac{1}{\lambda} \qquad \frac{d \ln g(x_n, \lambda)}{\lambda} = 0$$

$$-n + \sum_{i=1}^{n} x_i \,.\, \frac{1}{\lambda} = 0$$

$$\sum_{i=1}^{n} x_i \frac{1}{\lambda} = n ==> \left[n\lambda = \sum_{i=1}^{n} x_i \right] \div n$$

$$\lambda = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\lambda = \bar{x}$$

<u>Example</u>: Let x ~ Bernoulli Parameters (P) find Maximum likelihood estimator of P ?

<u>Solution</u>: Since x ~ Ber (P)

$$f(x) = \begin{cases} P^x (1-P)^{1-x} & \text{For x=0,1} \\ 0 & \text{Otherwise} \end{cases}$$

$$f(x_1, \dots, x_n, P) = f(x_1, P). f(x_2, P) \dots \dots f(x_n, P)$$

$$f(x_1, \dots., x_n, P) = P^{x_1}(1-P)^{1-x_1}. P^{x^2}(1-P)^{1-x_2} \dots.. P^{x_n}(1-P)^{1-x_n}$$

$$g(x_n, P) = P^{\sum_{i=1}^{n} x_i}(1-P)^{n-\sum_{i=1}^{n} x_i}$$

$$\text{Ln}\, g(x_n, P) = \ln\left[P^{\sum x_i}\right] + \ln\left[(1-P)^{n-\sum x_i}\right]$$

$$\ln g(x_n, P) = \sum_{i=1}^{n} x_i \ln(P) + (n - \sum_{i=1}^{n} x_i) \ln(1-P)$$

$$\frac{d \ln g(x_{n1}P)}{dP} = \sum_{i=1}^{n} x_i \frac{1}{P} + (n - \sum_{i=1}^{n} x_i) \frac{-1}{1-P} = \sum_{i=1}^{n} x_i \frac{1}{P} - (n - \sum_{i=1}^{n} x_i) \frac{1}{1-P}$$

$$\frac{d \ln g(x_n, P)}{dP} = \sum_{i=1}^{n} x_i \frac{1}{P} - (n - \sum_{i=1}^{n} x_i) \frac{1}{1-P}$$

$$\frac{d \ln g(x_n, P)}{dP} = 0$$

$$\sum_{i=1}^{n} x_i \frac{1}{P} - (n - \sum_{i=1}^{n} x_i) \frac{1}{1-P} = 0$$

$$\left[\sum x_i \frac{1}{P} = (n - \sum_{i=1}^{n} x_i)\frac{1}{1-P}\right] * \frac{1-P}{\sum_{i=1}^{n} x_i}$$

$$\frac{1-P}{P} = \frac{n - \sum x_i}{\sum x_i} - 1$$

$$\frac{1-P}{P} = \frac{P}{P} = \frac{n}{\sum_{i=1}^{n} x_i} - \frac{\sum x_i}{\sum x_i} ==> \ \frac{1}{P} - 1 = \frac{n}{\sum_{i=1}^{n} x_i} - 1$$

$$\frac{1}{P} = \frac{n}{\sum_{i=1}^{n} x_i} ==> \ \left[nP = \sum_{i=1}^{n} x_i\right] \div n$$

$$P = \frac{\sum_{i=1}^{n} x_i}{n} ==> P = \bar{\chi}$$

$$\hat{P} = \bar{\chi}$$

Example: Let $x_1 \ldots x_n$ ~ N(μ, σ^2) find Maximum-likelihood estimator of μ & σ^2?

Solution:

Since $x_1, \ldots., x_n$ ~ N(μ, σ^2) then $f(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1(x-\mu)^2}{2\sigma^2}} & \text{for } -\infty \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$

$$f(x_1, \ldots., x_n, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1(x_1-\mu)^2}{2\sigma^2}} . \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1(x_1-\mu)^2}{2\sigma^2}} \ldots \ldots \ldots \ n$$

$$g(x_1, \ldots., \mu, \sigma^2) = \frac{1}{\sigma^n \sqrt{(2\pi)^n}} e^{\frac{-1(\sum x_i - \mu)^2}{2\sigma^2}}$$

$$g(x_n, \mu, \sigma^2) = (2\pi\sigma^2)^{\frac{-n}{2}}\, e^{\frac{-1(\sum x_i - \mu)^2}{2}}$$

$$g(x_2, \mu, \sigma^2) = (2\pi\sigma^2)^{\frac{-n}{2}}\, e^{\frac{-1(\sum x_i - \mu)^2}{2\sigma^2}}$$

$$\ln g(x_n, \mu, \sigma^2) = (2\pi)^{\frac{-n}{2}} + \ln(\sigma^2)^{\frac{-n}{2}} + \ln e^{\frac{-(\sum x_i - \mu)^2}{2\sigma}}$$

$$\ln g(x_n, \mu, \sigma^2) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{(\sum x_i - \mu)^2}{2\sigma^2}$$

$$\frac{d \ln g(x_n, \mu, \sigma^2)}{d\mu} = 0 - 0 - \frac{1}{2\sigma^2}(-2)(\sum x_i - \mu)$$

$$\frac{d \ln g(x_n \mu)}{d\mu} = \frac{+2(2x_i - \mu)}{2\sigma^2}$$

$$\frac{(\sum x_i - \mu)}{\sigma^2} = 0 ==> (\sum x_i - \mu) = 0$$

$$\sum x_i = n.\mu ==> \ \mu = \frac{\sum x_i}{n} \qquad\qquad \mu = \bar{x} \ \rightarrow \hat{\mu} = \bar{x}$$

$$\ln g(x_n, \bar{x}, \sigma^2) = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{(\sum x_i - \bar{x})^2}{2\sigma^2}$$

$$\frac{d \ln g(x_n, \bar{x}, \sigma^2)}{d\sigma^2} = 0 - \frac{n}{2}.\frac{1}{\sigma^2} - \frac{0 - 2(\sum x_i - \bar{x})^2}{\mu\sigma^4}$$

$$\frac{d \ln g(x_n, \bar{x}, \sigma^2)}{d\sigma^2} = \ \frac{-n}{2\sigma^2} + \frac{(\sum x_i - \bar{x})^2}{2\sigma^4}$$

$$\frac{n}{2\sigma^2} + \frac{(\sum x_i - \bar{x})^2}{2\sigma^4}) = 0$$

$$\frac{(\sum x_i - \bar{x})^2}{2\sigma^4} = \frac{n}{2\sigma^2} ==> \left[2\sigma^2(\sum_{i=1}^{n} x_i - \bar{x}) = 2\sigma^4\, n\right] + 2\sigma^2 n$$

$$\frac{(\sum_{i=1}^{n} x_i - \bar{x})}{n} = \sigma^2 ==> \sigma^2 = S$$