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محاضرات الاحصاء ١

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The Unbiased Estimator

Let $X_1, X_2, \dots \dots, X_n$ br a random sample of size n from apopulation with probability density function $f(x{:}\,\theta)$. An estimator $\hat{\theta}$ of θ is a function of the random variables $X_1, X_2, \dots \dots, X_n$ which is free of the parameter θ .

An estimate is a realized value of an estimator that is obtained when a sample is actually taken .

Definition: An estimator θ^ of θ is said to be an unbiased estimator of θ if and only if

$$\boldsymbol{E}(\hat{\theta}) = \boldsymbol{\theta}$$

If $\theta^{\wedge}$ is not unbiased , then it is called a biased estimator of θ .

An estimator of a parameter may not equal to the actual to the actual value of the parameter for every realization of the sample $X_1, X_2, \dots, X_n$, but if it is unbiased then on an average it will equal to the parameter .

Example: Let $X_1, X_2, \dots \dots, X_n$ be a random sample from a normal population with mean μ and variance $\sigma^2 > 0$. Is the sample mean $\bar{X}$ an unbiased estimator of the parameter μ ?

Solution: Since , each $X_i \sim N(\mu, \sigma^2)$, we have $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

That is , the sample mean is normal with mean μ and variance $\frac{\sigma^2}{n}$.

Thus $E(\bar{X}) = \mu$. Therefore, the sample mean $\bar{X}$ is an unbiased estimator of μ

Example: Let $X_1, X_2, \dots \dots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2 > 0$,

Is the sample variance S^2 an unbiased estimator of the population variance σ^2 ?

Solution: Note that the distribution of the population is not given . However , we are given $E(\bar{X}) = \mu \; and \; E[(X_1 - \mu)^2] = \sigma^2$.

In order to find $E(S^2)$, we need find $E(\bar{X})$ and $E(\bar{X}^2)$. Thus we proceed to find these two expected values .

$$E(\bar{X}) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu$$

Similarly:

$$Var(\bar{X}) = Var\left(\frac{X_1 + X_2 + \cdots X_n}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}$$

Therefore

$$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2 = \frac{\sigma^2}{n} + \mu^2$$

Consider

$$E(S^2) = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n}(X_i^2 - 2\bar{X}X_i + \bar{X}^2)\right]$$

$$= \frac{1}{n-1} \mathrm{E}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$$

$$= \frac{1}{n-1}\left\{\sum_{i=1}^{n} E[X_i^2] - E[n\bar{X}^2]\right\}$$

$$= \frac{1}{n-1}\left[n(\sigma^2 + \mu^2) - n(\mu^2 + \frac{\sigma^2}{n})\right]$$

$$= \frac{1}{n-1}[(n-1)\sigma^2]$$

$$= \sigma^2 \quad .$$

Therefore , the sample variance S^2 is an unbiased estimator of the population variance σ^2 .

Example: If $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$ and let S_1^2, S_2^2 are estimators of σ^2 , Show that S_1^2 is unbiased estimators of σ^2 and S_2^2 is biosed estimator of σ^2. Such that :

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \ \ and \ \ S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Solation:

$$Z = \frac{(n-1)S^2}{\sigma^2} \sim X^2(n-1) \ \rightarrow E(Z) = (n-1)$$

$$\therefore Z_1 = \frac{(n-1)S_1^2}{\sigma^2} \sim X^2(n-1) \ \rightarrow E(Z_1) = (n-1) \ \_\_\_\_(1)$$

$$E(Z_1) = E\left(\frac{(n-1)S_1^2}{\sigma^2}\right) = \frac{(n-1)}{\sigma^2}E(S_1^2) \ \_\_\_\_\_(2)$$

From (1) and (2)

$$\frac{(n-1)}{\sigma^2}E(S_1^2) = (n-1) \ \rightarrow E(S_1^2) = \sigma^2$$

$\therefore S_1^2$ is an unbiased estimator of σ^2 .

$$Z_2 = \frac{nS_2^2}{\sigma^2} \sim \mathrm{X}^2(\mathrm{n}-1) \ \rightarrow \mathrm{E}(Z_2) = (n-1) \ \_\_\_\_(3)$$

$$E(Z_2) = E\left(\frac{nS_2^2}{\sigma^2}\right) = \frac{n}{\sigma^2}E(S_2^2) \ \_\_\_\_\_(4)$$

From (3) and (4) $\frac{n}{\sigma^2}E(S_2^2) = n-1 \ \rightarrow E(S_2^2) = \frac{(n-1)}{n}\sigma^2$

S_2^2 is on biased estimator of σ^2 .

Example: Let $X_1, X_2, \dots, X_n$ be a random sample from a benes popliation with parameter , show that $\bar{X}_n$ is an unbiased estimator.

Solation:

$$E(\bar{X}_n) = \frac{1}{n}E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}(E(X_i)) = \frac{1}{n}\left(\sum_{i=1}^{n} p\right)$$

$$= \frac{1}{p}np = p$$

Then $p^{\wedge} = \overline{X}_n$ is an unbiased estimator for p .

Example*:* Let $X_1, X_2 \; and \; X_3$ be a sample of size $n = 8$ from a distribution with un known mean $-\infty < \mu < \infty$ the variance σ^2 is a known positive number , show that both $\theta_1^{\wedge} = \bar{X}$ **and** $\theta_2^{\wedge} = \frac{1}{8}(2X_1 + X_2 + 5X_3)$ are unbiased estimator for μ . Compare the variance of $\theta_1^{\wedge}$ and $\theta_2^{\wedge}$

Solution :

$$E(\theta_1^{\wedge}) = E(\bar{X}) = E\left(\frac{1}{3}\sum_{i=1}^{3} X_i\right) = \frac{1}{3} \; 3\mu = \mu$$

$$E(\theta_2^{\wedge}) = \frac{1}{8}E(2X_1 + X_2 + 5X_3) = \frac{1}{8}[2E(X_1) + E(X_2) + 5E(X_3)]$$

$$= \frac{1}{8}(2\mu + \mu + 5\mu) = \frac{1}{8}(8\mu) = \mu$$

$\therefore \; \theta_1^{\wedge} \; , \theta_2^{\wedge}$ are unbiased estimators .

$$Var(\theta_1^{\wedge}) = V\left(\frac{1}{3}\sum_{i=1}^{3} X_i\right) = \frac{1}{9}[V(X_1) + V(X_2) + V(X_3)]$$

$$= \frac{1}{9}[\sigma^2 + \sigma^2 + \sigma^2] = \frac{1}{9} \; 3\sigma^2 = \frac{1}{3}\sigma^2$$

$$Var(\theta_2^{\wedge}) = V\left[\frac{1}{8}(2X_1 + X_2 + 5X_3)\right]$$

$$= \frac{1}{64}[V(2X_1 + X_2 + 5X_3)]$$

$$= \frac{1}{64}[4V(X_1) + V(X_2) + 25V(X_3)]$$

$$= \frac{1}{64}(4\sigma^2 + \sigma^2 + 25\sigma^2) = \frac{1}{64}(30\sigma^2)$$

$$= \frac{15}{32}\sigma^2$$

Factorization (jointly sufficient statistics)

Theorem : Let $X_1, X_2, \dots\dots, X_n$ be a random sample of size n from the density $f(.;\theta)$, where the parameter θ may be a vector . A set of statistics

$$S_1 = \sigma_1(X_1, X_2, \dots\dots, X_n), \dots\dots\dots\dots S_r = \sigma_r(X_1, X_2, \dots\dots, X_n).$$

Is jointly sufficient if and only if the joint density of $X_1, X_2, \dots\dots, X_n$ can be factored as $f_{X_1,\dots\dots X_n}(X_1, X_2, \dots\dots, X_n ;\theta)$

$$= g(\sigma_1(X_1, X_2, \dots\dots, X_n), \dots\dots, \sigma_r(X_1, X_2, \dots\dots, X_n);\theta)$$

$$= g(S_1, \dots\dots, S_r ;\ \theta)\ h(X_1, X_2, \dots\dots, X_n),$$

where the function $h(X_1, X_2, \dots\dots, X_n)$ is nonnegative and does not involve the parameter θ and the function $g(S_1, \dots\dots, S_r ;\ \theta)$ is nonnegative and depends on $(X_1, X_2, \dots\dots, X_n)$ only through the functions $\sigma_1(., \dots\dots, .), \dots., \sigma_r(., \dots., .)$.

Note that , according to Theorem . There are many possible sets of sufficient statistics. The above two theorems give us a relatively easy method for judging whether a certain statistic is sufficient or a set of statistics is jointly sufficient .

However , the method is not the complete answer since a particular statistic may be sufficient yet the user may not be clever enough to factor the joint density .

The theorems may also be useful in discovering sufficient statistics . Actually , the result of either of the above factorization theorems is intuitively evident if one notes the following:

1- If the joint density factors as indicated , then the likelihood function is proportional to $g(S_1, \dots\dots, S_r;\ \theta)$, which depends on the observations $X_1, \dots\dots, X_n$ only through $\sigma_1, \dots\dots, \sigma_r$ [the likelihood function is viewed as a function of θ , so $h(X_1, X_2, \dots\dots, X_n\ )$ is just a proportionality constant] , which means that the information about θ that the likelihood function contains is embodied in the statistics

$\sigma_1(., \dots\dots, .), \dots., \sigma_r(., \dots., .)$.

Example: $\sum_{i=1}^{n} X_i$ is sufficlent to μ , $X_i \sim (\mu, \sigma^2)$ by using faclorization theorm .

Solation:

$$f[X_1, \dots\dots, X_n, \mu] = f(X_1, \mu). f(X_2, \mu). \ \dots. \ . f(X_n, \mu)$$

Since $X_i \sim (\mu, \sigma^2)$

$$\therefore f(X) = \left\{ \frac{1}{\sigma\sqrt{2\pi}}\ e^{\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \right\} \quad \text{for} \quad -\infty < X < \infty$$

$$f[X_1, \dots\dots, X_n, \mu] = f(X_1, \mu). f(X_2, \mu). \ \dots. \ . f(X_n, \mu)$$

$$\sum \frac{1}{\sigma\sqrt{2\pi}}\ e^{\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} . \frac{1}{\sigma\sqrt{2\pi}}\ e^{\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} \ \dots. \ . \frac{1}{\sigma\sqrt{2\pi}}\ e^{\frac{-1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n . e^{\frac{-1(\sum X_i-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n . e^{\frac{[(\sum X_i)^2 - 2\mu\sum X_i + \mu^2]}{2\sigma^2}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n . \ e^{\frac{-(\sum X_i)^2}{2\sigma^2}} \ . e^{\frac{-(-2\mu\sum X_i + \mu)}{2\sigma^2}}$$

$$h(X) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \ . \ e^{\frac{-(\sum X_i)^2}{2\sigma^2}} \quad ,$$

$$g(t(X), \theta) = e^{\frac{-(-2\mu \sum X_i + \mu)}{2\sigma^2}}$$

$\therefore \sum X_i$ is sufficient statistic to μ .

<u>Example:</u> $\sum_i^n X_i$ is sufficient statistic to 1 $X \sim (1 , \sigma^2)$ by using faclorization theorem

<u>Solation:</u>

$$X_i \sim N(1 , \sigma^2)$$

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} \ . e^{\frac{-1(X-1)^2}{2\sigma^2}}$$

Since X_i is i.i.d

$$f(X_1, \dots, X_n , 1) = f(X_1, 1). f(X_2, 1). \ \dots \ . f(X_n, 1)$$

$$f(X_1, \dots, X_n , 1) = \frac{1}{\sigma\sqrt{2\pi}} \ . e^{\frac{-1(X-1)^2}{2\sigma^2}} \dots \frac{1}{\sigma\sqrt{2\pi}} \ . e^{\frac{-1(X-1)^2}{2\sigma^2}} .$$

$$f(X_1, \dots, X_n , 1) = \left[\frac{1}{\sigma\sqrt{2\pi}}\right]^n . e^{\frac{-1(\sum X_i - 1)^2}{2\sigma^2}}$$

$$f(X_1, \dots, X_n , 1) = \left[\frac{1}{\sigma\sqrt{2\pi}}\right]^n . e^{\frac{-1[(\sum X_i)^2 - 2\sum X_i + 1]}{2\sigma^2}}$$

$$f(X_1, \dots, X_n , 1) = \left[\frac{1}{\sigma\sqrt{2\pi}}\right]^n . e^{\frac{-1(\sum X_i)^2}{2\sigma^2}} . e^{\frac{-(-2\sum X_i + 1)}{2\sigma^2}}$$

$$h(X) = \left[\frac{1}{\sigma\sqrt{2\pi}}\right]^n . e^{\frac{-1(\sum X_i)^2}{2\sigma^2}} \quad ,$$

$$g(t(x), \theta) = . e^{\frac{-(-2\sum X_i + 1)}{2\sigma^2}}$$

$\therefore \sum_i^n X_i$ is sufficient statistic to 1 .

<u>Example</u>: $\boldsymbol{\sum_{i=1}^n X_i}$ is sufficient statistic to γ ,

$X_i \sim pio(\gamma)$ by using faclorization .

Solation:

Since $X_i \sim pio(\gamma)$

$$f(X) = \left\{ \frac{\gamma^X e^{-\gamma}}{X!} \quad for\ X = 0,1,,,,\infty \right\}$$

Since X_i is $i.i.d$

$$f[X_1, \dots\dots, X_n, \gamma] = f(X_1, \gamma).f(X_2, \gamma).\dots\ .f(X_n, \gamma)$$

$$f[X_1, \dots\dots, X_n, \gamma] = \frac{\gamma^{X_1} e^{-\gamma}}{X_1!} \ . \ \frac{\gamma^{X_2} e^{-\gamma}}{X_2!} \dots\dots\dots$$

$$f[X_1, \dots\dots, X_n, \gamma] = \frac{\gamma^{\sum X_i}\ e^{-\gamma}}{\sum_{i=1}^{n} X_i}$$

$$f[X_1, \dots\dots, X_n, \gamma] = \frac{1}{\sum_{i=1}^{n} X_i} \ . \left(\gamma^{\sum X_i}\ e^{-\gamma}\right)$$

$$h(X) = \frac{1}{\sum_{i=1}^{n} X_i} \quad ,$$

$$g(t(X), \theta) = \left(\gamma^{\sum X_i}\ e^{-\gamma}\right)$$

$\therefore$ $\sum_{i=1}^{n} X_i$ is sufficient statistic to γ.