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محاضرات الاحصاء ١

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Mean square error **متوسط مربعات الخطأ**

Definition:

The mean square error of the estimatorθ, denoted by MSE (θ) is defined as

MSE $(\hat{\theta})$ = E$(\hat{\theta}-\theta)^2$ = var $(\hat{\theta})$ + [E$(\hat{\theta})-\ \theta]^2$

Definition:

The unbiased estimator $\hat{\theta}$ that minimizes the mean square error is called the minimum variance unbiased estimator (MVUE) of θ.

Example:

Let X_1, X_2, X_3 be a sample of size n= 3 from a distribution with unknown mean μ, -∞< μ <∞, where the variance σ^2 is a known positive number.Show that both $\widehat{\theta_1}$ = $\bar{X}$ and $\widehat{\theta_2}$ = [(2X_1 + X_2 + 5X_3)/8] are unbiased estimators for μ. Compare the variances of $\widehat{\theta_1}$, and $\widehat{\theta_2}$.

Solution:

We have E$(\widehat{\theta_2})$ = E $(\bar{X})$ = $\frac{1}{3}$ 3 μ = μ. And E$(\widehat{\theta_2})$ = E = [(2X_1 + X_2 + 5X_3)/8]

= $\frac{1}{8}$ [(2E(X_1) + E(X_2) + 5E(X_3)] = $\frac{1}{8}$ [2μ + μ +5 μ) = μ

Hence, both 0, and 0, are unbiased estimators. However،

Var $(\widehat{\theta_1})$ = var $(\bar{X})$ = $\frac{1}{3}$ σ^2. Whereas Var $(\widehat{\theta_2})$ = var [(2X_1 + X_2 + 5X_3)/8]

= $\frac{1}{64}$ [4 var (X_1) + var (X_2) + 25 var (X_3) = $\frac{1}{64}$ 30 σ^2

Because var $(\widehat{\theta_1})$ < var $(\widehat{\theta_2})$, we see that $\bar{X}$ is a better unbiased estimator in the sense that the variance of $\bar{X}$ is smaller.

ملاحظات

- في حالة التقدير غير متحيز يكون $\left[E\left(\hat{\theta}\right)-\theta\right]=0$

وبالتالي فإن $MSE\left(\hat{\theta}\right)=V\left(\hat{\theta}\right)$

- نفرض أن $B\left(\hat{\theta}\right)=E\left(\hat{\theta}\right)-\theta$ بشكل عام (متحيز أو غير متحيز) $MSE\left(\hat{\theta}\right)=V\left(\hat{\theta}\right)+B\left(\hat{\theta}\right)$

Example:

If $x_1, \dots \dots \dots, x_n \sim N(M, \sigma^2)$ consider the two estimators of σ^2, $\widehat{\theta_1} = {s_1}^2 = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$, $\widehat{\theta_2} = {s_2}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$. Find the $e(\theta_1, \theta_2)$.

Solution :

$$E({s_1}^2) = \sigma^2 \;\Rightarrow MSE(s^2) = var\,(s^2)$$

$$v\left(\frac{(n-1)s^2}{\sigma^2}\right) = 2(n-1) \;\Rightarrow\; \frac{(n-1)^2}{\sigma^4}\, v(s^2) = 2(n-1) \Rightarrow v(s^2)$$
$$= \frac{2\sigma^4}{(n-1)} = MSE\,({s_1}^2)$$

للتوضيح

$$v({s_2}^2) = \frac{2(n-1)}{n^2}\sigma^4 \qquad , \quad E({s_2}^2) = \frac{(n-1)}{n}\sigma^2$$

$$B({s_2}^2) = E({s_2}^2) - \sigma^2 = \frac{(n-1)}{n}\sigma^2 - \sigma^2 = \sigma^2 - \frac{1}{n}\sigma^2 - \sigma^2 = -\frac{1}{n}\sigma^2$$

$$MSE({s_2}^2) = v({s_2}^2) + B({s_2}^2) = \frac{(2n-2)\sigma^4}{n^2} + \frac{1}{n^2}\sigma^4 = \frac{(2n-2+1)\sigma^4}{n^2}$$
$$= \frac{(2n-1)\sigma^4}{n^2}$$

$$e = \frac{MSE({s_2}^2)}{MSE({s_1}^2)} = \frac{\frac{(2n-1)}{n^2}\sigma^4}{\frac{2}{(n-1)}\sigma^4} = \frac{(2n-1)(n-1)}{2n^2} < 1$$

${s_2}^2$ is relatively more efficient than s^2.

Definition:

اذا كان $\hat{\theta}$ تقدير فير محيز لـ θ وكان $v(\hat{\theta}) = \frac{1}{nE\left[-\frac{\partial^2 \ln(f)}{\partial \theta^2}\right]}$، يكون $\hat{\theta}$ التقدير الغير محيز ذو أقل تباين المنتظم ويرمز له (Uniformly Minimum Variance Unbiased Estimator) (UMVUE)

Example: let $x_1, \dots \dots \dots, x_n \sim N(M, \sigma^2)$ show that $\bar{x}$ is an efficient est.

Solution :

$$f_{(x)} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\ln(f) = \ln\left(\frac{1}{\sqrt{2\pi}\;\sigma}\right) - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{\partial \ln(f)}{\partial \mu} = 0 - \frac{1}{2\sigma^2}\,2(x-\mu)\,(-1) = \frac{(x-\mu)}{\sigma^2} = \frac{x}{\sigma^2} - \frac{\mu}{\sigma^2}$$

$$\frac{\partial^2 \ln(f)}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

$$\frac{1}{nE\left[-\frac{\partial^2 \ln(f)}{\partial \mu^2}\right]} = \frac{1}{nE\left[-\frac{1}{\sigma^2}\right]} = \frac{1}{n\frac{1}{\sigma^2}} = \frac{\sigma^2}{n} = v(\bar{x})$$

$\bar{x}$ is an efficient estimator of μ

$\bar{x}$ is an UMVUE of μ

- ❖ اذا كان التقديرين غير متحيز يكون $e(\theta_1, \theta_2) = \frac{v(\theta_2)}{v(\theta_1)}$
- ❖ اذا كان التقديرين بشكل عام سواء (متحيز أو غير متحيز) نستخدم القانون $e(\theta_1, \theta_2) = \frac{MSE\ (\theta_2)}{MSE\ (\theta_1)}$
- ❖ حل الأمثلة لتقدير المتسق بالطريقة الثانية.

Example: let $x_1, \dots \dots \dots, x_n \sim Po(\lambda)$ show that $\overline{x_n}$ is an consis est. of the (λ) .

Solution :

1) $E(\bar{x}) = \frac{1}{n}\big(E(x_1) + \cdots + E(x_n)\big) = \frac{1}{n}(\lambda + \cdots + \lambda)$

$$\mathrm{n-times} \qquad = \frac{1}{n}(n\lambda) = \lambda$$

2) $v(\bar{x}) = \frac{\lambda}{n}$

$$\lim_{n\to\sigma} v(\bar{x}) = \lim_{n\to\sigma} \frac{\lambda}{n} = 0$$

$\bar{x}$ is a consistent est of λ.

Example: let $x_1, \dots\dots\dots, x_n \sim N(\mu, \sigma^2)$

a) show that the sample variance s^2 is a consistent estimator for σ^2.
b) Show that the max . liklelihood estimator for μ & σ^2 are consistent estimator for μ & σ^2

Solution :

a)

1) $E(s^2) = \sigma^2$

2) $v(s^2) = \frac{2\sigma^4}{n-1}$

$$\lim_{n\to\sigma} v(s^2) = \lim_{n\to\sigma} \frac{2\sigma^4}{n-1} = 0$$

s^2 is consistent estimator of σ^2

b)

MLE $\hat{\mu}=\bar{X}$ & MLE $\sigma^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$

1) $\mathrm{E}(\bar{X}) = \mu$
2) $\mathrm{V}(\sigma^2) = \frac{\sigma^2}{n}$ $\Rightarrow$ $\lim_{n\to\infty} v(\bar{X}) = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$

$\bar{X}$ is consistent est. of μ

MLE $\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$

$\mathrm{E}(\hat{\sigma}^2) = \mathrm{E}(\frac{1}{n}\sum(X_i - \bar{X})^2) = \frac{(n-1)}{n}\left[E\,\frac{\sum(X_i - \bar{X})^2}{(n-1)}\right] = \frac{(n-1)}{n}\sigma^2$

$\therefore \sigma^2$ is biased

$$Z = \frac{(n-1)}{\sigma^2} S^2 \sim X^2(n-1)$$

$$E(Z) = (n-1)$$

$$V(Z) = 2(n-1)$$

$$B(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = \left(1 - \frac{1}{n}\right)\sigma^2 - \sigma^2$$

$$= \sigma^2 - \frac{1}{n}\sigma^2 - \sigma^2 \quad = -\frac{1}{n}\ \sigma^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2 = \frac{(n-1)}{n}\left[\frac{1}{(n-1)}\sum(X_i - \bar{X})^2\right] = \frac{(n-1)}{n}S^2$$

$$V(\hat{\sigma}^2) = V\left[\frac{(n-1)}{n}S^2\right] = \frac{(n-1)^2}{n^2}\ \mathrm{V}\ (S^2) = \frac{(n-1)^2}{n^2}\ \frac{2\sigma^4}{(n-1)}$$

$$= \frac{2(n-1)\sigma^4}{n^2}$$

$$\lim_{n\to\infty} B(\hat{\sigma}^2) = \lim_{n\to\infty}\ \frac{-\sigma^2}{n} = 0$$

$$\lim_{n\to\infty} V\ (\hat{\sigma}^2) = \lim_{n\to\infty}\ \frac{2(n-1)(\sigma^4)}{n^2} = 0$$

$$\therefore \mathrm{E}\ [(\theta^n - \theta)^2]\ = \mathrm{V}\ (\hat{\theta}) + \left[B\ \left(\hat{\theta}\right)\right]^2$$

$$\therefore \lim_{n\to\infty} E\ \ (\hat{\sigma}^2 - \sigma^2\ )^2 = \lim_{n\to\infty} V\ (\hat{\sigma})^2 + \lim_{n\to\infty}[B\ (\hat{\sigma}^2)]^2$$

$$= 0 + 0 = 0$$

Sufficiency الكفاية

In the statistical inference problems on a parameter, one of the major questions is: Can a specific statistic replace the entire data without losing pertinent information?

Suppose X_1 ,…, X_n is random sample from a probability distribution with unknown parameter θ. In general, statisticians look for ways of reducing a set of data so that these data can be more easily understood without losing the meaning associated with the entire collection of observations. Intuitively, a statistic U is a sufficient statistic for a parameter θ if U contains all the information available in the data about the value of θ.

For example, the sample mean may contain all the relevant information about the parameter μ, and in that case U = $\bar{X}$ is called a sufficient statistic for μ . An estimator that is a function of a sufficient statistic can be deemed to be a "good" estimator, because it depends on fewer data values. When we have a sufficient statistic U for , we need to concentrate only on U because it exhausts all the information that the sample has about θ. That is, knowledge of the actual n observations does not contribute anything more to the inference about θ.

Definition :

Let X_1,…., X_n be a random sample from a probability distribution with unknown parameter θ .Then, the statistic U=g (X_1,…., X_n) is said sufficient for θ. if the conditional pdf or pf of X_1,…., X_n given U = u does not depend on θ for any value of u. An estimator of θ that is a function of a sufficient statistic for θ is said to be a sufficient estimator of θ.

Definition: Simple consistency

Let $T_1, T_2, \dots, T_n$ be a sequence of estimators of $\tau(\boldsymbol{\theta})$, where $T_n = t_n$ (X_1, ….,X_n). The sequence $\{T_n\}$ is defined to be a simple (or weakly) consistent sequence of estimators of τ ($\boldsymbol{\theta}$) if for every ε> 0 the following is satisfied:

$$\lim_{n\to\infty} P_{\boldsymbol{\theta}}[\tau(\theta) - \varepsilon < T_n < \tau(\boldsymbol{\theta}) + \varepsilon]$$

Remark: If an estimator is a mean-squared-error consistent estimator, it is also a simple consistent estimator, but not necessarily vice versa.

Proof :

$P_{\boldsymbol{\theta}}[\tau(\boldsymbol{\theta}) - \varepsilon < T_n < \tau(\boldsymbol{\theta}) + \varepsilon]$= P $[|T_n - \tau(\boldsymbol{\theta})| < \varepsilon]$

= $P_{\boldsymbol{\theta}}$ $[[T_n - \tau(\boldsymbol{\theta})]^2 < \varepsilon^2] \geq 1 - \frac{\mathcal{E}_{\boldsymbol{\theta}}\,[[T_n - \tau(\boldsymbol{\theta})]^2]}{\varepsilon^2}$

by the Chebyshev inequality. As n approaches infinity,$\mathcal{E}_{\boldsymbol{\theta}}$ $[[T_n - \tau(\boldsymbol{\theta})]^2]$ approaches 0. Hence lim $P_{\boldsymbol{\theta}}$ $[\tau(\boldsymbol{\theta})$ – ε < T_n, < $\tau(\boldsymbol{\theta})$ + ε] = 1

Example:

Let $x_1, \dots\dots, x_n$ be iid Bernoulli random variables with parameter θ. show that$\sum_{i=1}^{n} x_i$ is sufficient for θ.

Solution:

The joint probability mass function of $x_1, \dots\dots, x_n$ is

$$f(x_1, \dots\dots, x_n; \theta) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

Because $U = \sum_{i=1}^{n} x_i$ we have

$$f(x_1, \dots\dots, x_n; \theta) = \theta^{U} (1-\theta)^{n-U} \qquad ,0 \leq U \leq n$$

Also, because $U \sim b(n\,, \theta)$ we have

$$f(u,\theta)=\binom{n}{u}\theta^{U}\,(1-\theta)^{n-U} \qquad ,0\le U\le n$$

Also,

$$f(x_1,\dots,x_n|U=u)=\frac{f(x_1,\dots,x_n;u)}{f_U(u)}=\left\{\begin{matrix}\frac{f(x_1,\dots,x_n)}{f_U(u)} & u=\sum_{i=1}^{n}x_i\\ 0 & o.w.\end{matrix}\right\}$$

Therefore ,

$$f(x_1,\dots,x_n|U=u)=\frac{f(x_1,\dots,x_n;u)}{f_U(u)}=\left\{\begin{matrix}\frac{\theta^u(1-\theta)^{n-u}}{\binom{n}{u}\theta^u(1-\theta)^{n-u}}=\frac{1}{\binom{n}{u}} & u=\sum_{i=1}^{n}x_i\\ 0 & o.w.\end{matrix}\right\}$$

Which is independent of . Therefore U is sufficient for .

Example:

let $x_1,\dots\dots,x_n$ be arandom sample from passion (λ) show that the mean $\bar{x}$ is consistent to λ

Solution:

$x_i\sim$ piosson Distribution

$$v(\bar{x})=v\left[\sum\frac{x_i}{n}\right]\ \Rightarrow\ \frac{1}{n^2}v\left[\sum x_i\right]=\frac{1}{n^2}v[x_1+x_2+\cdots+x_n]$$
$$=\frac{1}{n^2}[\lambda+\lambda+\cdots]==\frac{1}{n^2}n\lambda$$

$v(\bar{x})=\frac{\lambda}{n}$ where $\epsilon=k\frac{\sigma}{x}=k\sqrt{\frac{\lambda}{n}}\quad\Rightarrow\quad k=\frac{\epsilon\sqrt{n}}{\sqrt{\lambda}}\quad\Rightarrow\quad k^2=\frac{\epsilon^2\,n}{\lambda}$

$$P\left\{|\bar{x}-\lambda|>k\sqrt{\frac{\lambda}{n}}\right\}\le\frac{1}{\frac{\epsilon^2\,n}{\lambda}}=\frac{\lambda}{\epsilon^2\,n}$$

$$\lim_{n\to\infty}P\left\{|\bar{x}-\lambda|>k\sqrt{\frac{\lambda}{n}}\right\}\le\frac{\lambda}{\epsilon^2\,n}\quad by\ chebysheos=0$$

$\lim_{n\to\infty}\left[\frac{\lambda}{\epsilon^2\,n}\right]=\frac{1}{\infty}=0$ then $\bar{x}$ is consistent to λ

Example:

let $x_1, \dots \dots, x_n$ be arandom sample from $N(\mu, \sigma)$

$S_n{}^2 = \sum \left[\frac{x_i - \bar{x}}{n-1}\right]^2$ show that $S_n{}^2$ is consistent to σ^2

Solution:

Since $\frac{(n-1)}{\sigma^2} S_n{}^2 \sigma \grave{X}_{(n-1)}$ then $v(S^2) = 2r \quad since\ r = n-1$

$$v(S_n{}^2) = 2(n-1)$$

$$v\left[\frac{n-1}{\sigma^2} S_n{}^2\right] = 2(n-1)$$

$$\left[\frac{(n-1)^2}{\sigma^4} v(S_n{}^2) = 2\,(n-1)\right] * \frac{\sigma^4}{(n-1)^2}$$

$v(S^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} \Rightarrow v(S^2) = \frac{2\sigma^4}{(n-1)}$ where $\epsilon = k\sigma_{S_n}$

$$\epsilon = k\sqrt{\frac{2\sigma^4}{(n-1)}} \quad \Rightarrow \quad k = \frac{\epsilon\sqrt{n-1}}{\sqrt{2\sigma^2}} \quad \Rightarrow \quad k^2 = \frac{\epsilon^2\,(n-1)}{2\sigma^4}$$

$$\lim_{n\to\infty} \left\{ \left|S_n{}^2 - \sigma^2\right| > k\sqrt{\frac{2\sigma^4}{(n-1)}} \right\} \leq \frac{1}{\frac{\epsilon^2\,(n-1)}{2\sigma^4}}$$

$$\lim_{n\to\infty} \left\{ \left|S_n{}^2 - \sigma^2\right| > k\sqrt{\frac{2\sigma^4}{(n-1)}} \right\} \leq \frac{2\sigma^4}{\epsilon^2\,(n-1)} \qquad by\ chebysheos = 0$$

$$\frac{2\sigma^4}{\infty} = 0$$

$S_n{}^2$ is consistent to σ^2.