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محاضرات الاحصاء ١

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Roa – Black well theorem

If $\hat{\theta}$ is unbiased est . for θ and t(x) is sufficient for θ, then the estimation $\bar{\theta}$ where

$$\bar{\theta} = E\left[\frac{\hat{\theta}}{t(x)}\right]$$

is also unbiased and its variance less than or equal to the variance of $\hat{\theta}$ i .e :

$$v(\bar{\theta}) \leq v(\hat{\theta})$$

Ex : $x_1 , x_2, \dots , x_n$is ar.s from Ber(θ) if $x_1$is unbiased est for θ, Find a better est then By using the Roa_Black well Theorem .

Sol :

$$f(x,\theta) = \theta^x(1-\theta)^{1-x}$$

$$f(x_1,x_2,\dots,x_n) = \prod_{i=1}^{n} f(x_i,\theta)$$

$$= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}.1$$

h(x) does not depend upon θ

Then $\sum x_i$ is s.s for θ

We have $E(x_1) = \theta$

By using the Roa_Black well theorem we get

$\bar{\theta} = E\left(\frac{\hat{\theta}}{t(x)}\right) = E\left(\frac{x_1}{\sum x_i}\right)$ is better est than x_1

Now what is $\bar{\theta}$

$$E\left(\frac{x_1}{\sum x_i}\right) = E\left[\frac{x_1 = x_i}{\sum x_i = t}\right]$$

$x_1 = x_i$ is unbiased and $\sum x_i = t$ is sufficient

$$= \sum_{x=0,1} x_1 P\left[\frac{x_1 = x_i}{\sum x_i = t}\right]$$

$$= 0.P\left[\frac{x_1 = 0}{\sum x_i = t}\right] + 1.P\left[\frac{x_1 = 1}{\sum x_i = t}\right]$$

$$P\left[\frac{x_1 = 1}{\sum x_i = t}\right]$$

$$= \frac{P[x_1 \,,\, \sum_{i=1}^{n} x_i = t]}{p\left(\sum_{i=1}^{n} x_i = t\right)}$$

$$P[x_1 = 1] = \theta$$

$$x \sim Ber(\theta)$$

$$\sum x_i \sim Bin\ (n, \theta)$$

$$p\left(\sum x_i = t\right) = \binom{n}{t}\theta^t(1-\theta)^{n-t}$$

$$\sum x_i \sim Bin(n-1, \theta)$$

$$P\left[\sum x_i = t-1\right] = \binom{n-1}{t-1}\theta^{t-1}(1-\theta)^{n-t}$$

$$\rightarrow \frac{P[x_1 = 1] * P[\sum_{i=2}^{n} x_i = t-1]}{P[\sum x_i = t]}$$

$$= \frac{\theta\binom{n-1}{t-1}\theta^{t-1}(1-\theta)^{n-t}}{\binom{n}{t}\theta^t(1-\theta)^{n-t}}$$

$$= \frac{\binom{n-1}{t-1}}{\binom{n}{t}} = \frac{\frac{(n-1)!}{(t-1)!\left((n-1)-(t-1)\right)!}}{\frac{n!}{t!\,(n-t)!}}$$

$$\frac{(n-1)!}{(t-1)!\,(n-t)!} - \frac{t!\,(n-t)!}{n!}$$

$$\frac{(n-1)!}{(t-1)!} * \frac{t_1(t-1)!}{n(n-1)!} = \frac{t}{n} = \frac{\sum x_i}{n} = \bar{x}$$

$\therefore$ $\bar{x}$ is a better estimator than $x_1$for θ

Example:

Let $x_1, x_2, \ldots, x_n \sim P(\theta) \ \rightarrow iid$ use the Roa – Black well theorem to find an estimator for θ better than x_1

iid= identically independent distribution

Solution:

$$f(x_1, \theta) = \frac{\theta^x e^{-\theta}}{x!}$$

$$f(x_1, x_2, \ldots, x_n) = \frac{n\theta^{\sum x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!} = n\theta^{\sum x_i} e^{-n\theta} * \frac{1}{\prod_{i=1}^{n} x_i!}$$

$\therefore \sum x_i$ is s.s for

Now , to Find $\bar{\theta}$

$$\bar{\theta} = E\left[\frac{x_1}{t(x)} = \sum_{i=1}^{n} x_i = t\right]$$

$$P\left[x_1 = \frac{X_1}{\sum x_i = t}\right] = \frac{P[x_1 = X_1, \sum_{i=1}^{n} x_i = t]}{P[\sum_{i=1}^{n} x_i =]}$$

$$= \frac{p[x_1 = X_1, \sum x_i = t - x_1]}{p[\sum x_i = t]} = \frac{p[x_1 = X_1, \sum x_i = t]}{P[\sum x_i = t]}$$

$$P[x_1 = X_1] = \frac{\theta^{x_1} e^{-\theta}}{x_1!}$$

$$x \sim p(\theta)$$

$$\sum x_i \sim P(n\theta), \sum x_i \sim P((n-1)\theta)$$

$$P\left[\sum x_i = t\right] = \frac{(n\theta)^t e^{-n\theta}}{t!}$$

$$P\left[\sum x_i, t - x_1\right] = \frac{\left((n-1)\theta\right)^{t-x_1}}{(t-x_1)!}$$

$$P\left[\frac{x_1 = X_1}{\sum x_i = t}\right] = \frac{\dfrac{\theta^{x_1} e^{-\theta}}{x_i!} \; \dfrac{(n-1)\theta^{t-x_1} e^{-(n-1)\theta}}{(t-x)!}}{\dfrac{(n\theta)^t \, e^{-n\theta}}{t!}}$$

$$\frac{\theta^{x_1} e^{-\theta}}{x_1!} * \frac{(n-1)\theta^{t-x_1} e^{-(n-1)\theta}}{(t-x)!} * \frac{t!}{(n\theta)^t \, e^{-n\theta}}$$

$$\frac{t!\,(n-1)^{t-x_1}}{x_1!\,(t-x_1)!\,n^t}$$

Now $n^t = n^{x_1} * n^{t-x_1}$

$$P\left[\frac{x_1 = X_1}{\sum x_i = t}\right] = \frac{t_i (n-1)^{t-x_1}}{x_1!\,(t-x_1)!\,n^{x_1} * n^{t-x_1}}$$

$$= \frac{t!}{x_1!\,(t-x_1)!} * \left(\frac{1}{n}\right)^{x_1} \left(\frac{n-1}{n}\right)^{t-x_1}$$

$$= \frac{t!}{x_1!\,(t-x_1)!} \left(\frac{1}{n}\right)^{x_1} \left(1 - \frac{1}{n}\right)^{t-x_1}$$

$$\binom{t}{x_1} \left(\frac{1}{n}\right)^{x_1} \left(1 - \frac{1}{n}\right)^{t-x_1} \sim Bin\left(t, \frac{1}{n}\right)$$

$$E\left[\frac{x_1}{\sum x_i}\right] = t * \frac{1}{n} = \frac{t}{n} = \frac{\sum x_i}{n} = \bar{x}$$

$\bar{x}$ is a better estimator for θ

C**ompleteness :-** A statistic t(x) is said to be complete if for all θ the function h(t) statistic $E(h(t) = 0$ which implies that $h(T) = 0$

Ex: let $x \sim Ber(\theta)$show that x is complete

Sol :

$$f(x, \theta) = \theta^x (1-\theta)^{1-x}$$

We have

$$E(h(x) = 0 \text{ we prove } h(x) = 0$$

$$E\left(h(x)\right) = \sum_{x=0,1} h(x) * f(x,\theta) = 0$$

$$h(0) * f(0,\theta) + h(1) * f(1,\theta) = 0$$

$$h(0) * (1-\theta) + h(1) * \theta = 0$$

$$h(0) - h(0) * \theta + h(1) * \theta = 0$$

$$h(0) + \theta\left(h(1) - h(0)\right) = 0$$

$\theta \neq 0$ is perimeter

$$h(1) - h(0) = 0 \rightarrow h(1) = h(0)$$

$$\because h(0) = 0$$

$$h(1) = 0 \qquad x = 0,1$$

$$\therefore x\ is\ complete$$

Example:

$x_1, x_2, \dots, x_n$ is ar.s from a dist $Ber(\theta)$. show that $T = \sum x_1$ is complete sufficient statistic for θ

Solution:

$$f(x,\theta) = \theta^x(1-\theta)^{1-x}$$

$$\prod_{i=1}^{n} f(x,\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} * 1$$

$$\therefore \sum x_i\ \ is\ s.s\ \ for\ \theta$$

Now , we went to prove $T = \sum x_i$ is C.S.S

$$x \sim Ber(\theta)$$

$$\sum x_i \sim Bin\ (n,\theta)$$

$$E\left(h(t)\right) = 0$$

$$E[h(T)] = \sum_{T=o}^{n} h(T) * f(T,\theta) = 0$$

$$f(T,\theta) = f\left(T = \sum x_i, \theta\right) = \binom{n}{T} * \theta^T (1-\theta)^{n-T}$$

$$E\big(h(T)\big) = h(0)\binom{n}{0}\theta^0(1-\theta)^{n-0} + h(1)\binom{n}{1}\theta(1-\theta)^{n-1} + \cdots$$

$$h(n)\binom{n}{n}\theta^n(1-\theta)^{n-n} = 0$$

$$h(0)(1-\theta)^n = 0 \ \ \%(1-\theta)^n$$

$$h(0) = 0$$

$$h(1)\binom{n}{1}\theta(1-\theta)^{n-1} = 0 \ \ \%\binom{n}{1}\theta(1-\theta)^{n-1}$$

$$h(1) = 0$$

$$h(0) = h(1) = \cdots = h(n) = 0$$

$$h(T) = 0\ , T = 1,2,\ldots,n$$

$$\sum x_i \ \ is\ C.S.S\ for\ \theta$$

Exponential Family of distribution

<u>Definition</u>: A one Parameter exponential family of distribution is that if $f(x,\theta)$ can be express in the from

$$f(x,\theta) = a(\theta) * b(x)e^{c(\theta)dx}$$

$$or\ f(x,\theta) = e^{c(\theta)dx} + b(x) + a(\theta) \quad \alpha < x < \beta$$

Where α , β does not depot upon θ.

<u>Example:</u> if $x \sim Ber(\theta)$, show that $f(x,\theta)$. belongs to exponential family

$$f(x,\theta) = \theta^x (1-\theta)^{1-x}$$

$$= \theta^x(1-\theta)(1-\theta)^{-x}$$

$$= \theta^x(1-\theta) * \frac{1}{(1-\theta)^x}$$

$$= (1-\theta)\left(\frac{\theta}{1-\theta}\right)^{x}$$

$$= (1-\theta)e^{\ln\left(\frac{\theta}{1-\theta}\right)^{x}}$$

$$= (1-\theta)e^{x\ln\left(\frac{\theta}{1-\theta}\right)}$$

$$a(\theta) = (1-\theta), b(x) = 1, c(\theta) = \ln\left(\frac{\theta}{1-\theta}\right), d(x) = x$$

$f(x,\theta)$ belongs to exponential family.

H.w : $x \sim P(\theta)$ show that $f(x,\theta)$ belong to exponential family .

Theorem :

Let $f(x,\theta)$ be a P.d.f which represent a regular case of the exponential class . Than if $x_1, x_2, \dots, x_n$.

Where (n) is a fixed positive integer is a random sample from a dist , with P. d .f $f(x,\theta)$ the statistic $t = \sum_{i=1}^{n} di$ is sufficient statistic for θ and the family $g(t,\theta)$ of probability density family of t is complete that is t is C.S.S for θ.

Theorem : Any function of C.S.S is MVUE of it expectation

<u>Example</u>: if $x \sim \exp(\theta)$ find MVUE

$$f(x,\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$$

$$a(\theta) = \frac{1}{\theta} \ , b(x) = 1 \ , c(\theta) = -\frac{1}{\theta} \ , d(x) = x$$

$\mathrm{f(x,\theta)} =$ belong to exponential family $\mathrm{t} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d(x_i)} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x_i}$ is C.S.S for θ .