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Best Critical Region

We consider the properties a satisfactory hypothesis test (or critical region) should possess. To introduce our investigation, we begin with a non-statistical example.

Example: Say that you have α dollars with which to buy books. Further, suppose that you are not interested in the books themselves, but only in filling as much of your book- shelves as possible. How do you decide which books to buy? Does the following approach seem reasonable? First of all, take all the available free books. Then start choosing those books for which the cost of filling an inch of bookshelf is smallest. That is, choose those books for which the ratio c/w is a minimum, where w is the width of the book in inches and c is the cost of the book. Continue choosing books this way until you have spent the α dollars.

Definition: uniformly most powerful critical region of size α .

A test defined by a critical region C of size α is a uniformly most powerful test if it is a most powerful test against each simple alternative in H_1 . The critical region C is called a uniformly most powerful critical region of size α.

LIKELHOOD RATIO TESTS

We consider a general test – construction method that is applicable when either of both of the null and alternative hypotheses – say . H_0 and H_1-are composite . We continue to assume that the functional form of the p. d. f. is Known . but that it depends on one or more unknown parameters . that is we assume that the p. d. f. of X is $f(X:\theta)$, where θ represent one or more inknown parameters . we let Ω denote the total parameters space – that is , the set of all possible values of the parameter θ given by either H_0or H_1 . these hypotheses will be stated as .

$H_0: \theta \in \omega, \qquad H_1: \theta \in \omega'.$

Where ω is a subset of Ω and ω′ is the complement of ω with respect to Ω . the test will be constructed with the use of a ratio of likelihood functions that have been maximized in ω and Ω , respectively . n a sense , this is natural generalization of the ratio appearing in the Neyman – Pearson lemma when the two hypotheses were simple .

Definition:

The likelihood ratio is the quotient

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

Where$L(\widehat{\omega})$ is the maximum of the likelihood function with respect to θ when $\theta \in \omega$ and $L(\widehat{\Omega})$ id the maximum of the likelihood function with respect to θ when $\theta \in \Omega$

Definition:

Consider the test of the simple null hypothesis $Ho: \theta = \theta o$ against the simple alternative hypothesis $H_1: \theta = \theta_1$ Let C be a critical region of size α: that is , $\alpha = P(c: \theta o)$. Then C is a best critical region of size α if. For every other critical region D of size $\alpha = P(D: \theta o)$. We have $P(C: \theta_1) \geq P(D: \theta_1)$

That is. When $H_{1:}\theta = \theta_1$is true . the probability of rejecting H_0: $\theta = \theta_0$ with the use of the critical region C is at least as great as the corresponding probability with the use of any other critical region D of size α .

Thus a best critical region of size α is the critical region that has great- est power among all critical region of size α . the Neyman- Person lemma gives sufficient conditions for a best critical region of size α

Theorem:(Neyman- Person lemma)

Let $x_1 . x_2 \dots \dots . x_n$ be a random sample of size n from a distribution with pdf or pmf $f\ (x: \theta)$. where θ_0 and θ_1 are two possible values of θ .

$$L\ (\theta) = L\ (\theta: x_1 . x_2 \dots . . x_n) = f(x_1: \theta)\ f\ (x_2: \theta) \dots f(x_{n:}\theta).$$

If there exist a positive constant k and a subset C of the sample space such that

(a) $P[(x_1 . x_2 \dots . . x_n) \in C: \theta_0] = \alpha.$

(b) $\frac{L(\theta_0)}{L(\theta_1)} \leq k\ for\ (x_1 . x_2 \dots . . x_n) \in C. and$

(c) $\frac{L(\theta_0)}{L(\theta_1)} \geq k\ for\ (x_1 . x_2 \dots . . x_n) \in C.$

Then C is a best critical region of size α for testing the simple null hypothesis $Ho: \theta = \theta o$ against the simple alternative hypothesis $H_1: \theta = \theta_1$

Proof: We prove the theorem when the random variables are the continuous type: for discrete – type random variables replace the integral signs by summation signs To simplify the exposition . we shall use the following notation :

$$\int_B L(\theta) = \int B^{\cdots} \int \; L(\theta: x_1\, x_2 \ldots \ldots x_n)\, dx_1\, dx_2 \ldots .\, dx_n.$$

Assume that there exists another critical region of size α- say . D, such that. in this new notation

$$a = \int_c L(\theta_0) = \int_D L(\theta_0).$$

Then we have

$$0 = \int_c L(\theta_0) - \int_D L(\theta_0)$$

$$= \int_{C \cap D} L(\theta_0) + \int_{C \cap D} L(\theta_0) - \int_{C \cap D} L(\theta_0) - \int_{C \cap D} L(\theta_0).$$

Hence

$$0 = \int_{C \cap D} L(\theta_0) - \int_{C \cap D} L(\theta_0)$$

By hypothesis (b). $\mathrm{k}\mathrm{L}(\theta_1) \geq \mathrm{L}(\theta_0)$ at each point in C and therefore in $C \cap D$, thus .

$$k \int_{C \cap D} L(\theta_1) \geq \int_{C \cap D} L(\theta_0)$$

By hypothesis (b). $\mathrm{k}\mathrm{L}(\theta_1) \leq \mathrm{L}(\theta_0)$ at each point in C and therefore in $C \cap D$: thus we obtain .

$$k \int_{C \cap D} L(\theta_1) \leq \int_{C \cap D} L(\theta_0)$$

Consequently .

$$0 = \int_{C \cap D} L(\theta_0) - \int_{C \cap D} L(\theta_0) \leq (k) \left\{ \int_{C \cap D} L(\theta_1) - \int_{C \cap D} L(\theta_1) \right\}$$

That is .

$$0 \leq (k) \left\{ \int_{C \cap D} L(\theta_1) + \int_{C \cap D} L(\theta_1) - \int_{C \cap D} L(\theta_1) - \int_{C \cap D} L(\theta_1) \right\}$$

Or equivalently .

$$0 \leq (k) \left\{ \int_c L(\theta_1) - \int_D L(\theta_1) \right\}$$

Thus.

$$\int_c L(\theta_1) \geq \int_D L(\theta_1):$$

That is . $P(C: \theta_1) \geq P(D: \theta_1)$ since that is true for rvery critical region D of size α . C is a best critical region of size α.

Example:

Let $X_1, X_2 \ldots \ldots X_n$ denote a random sample of size n from a Poisson distribution with mean λ A best critical region for $H_0: \lambda = 2$ against $H_1: \lambda = 5$ given by

$$\frac{L(2)}{L(5)} = \frac{2^{\sum_{i=1}^{n} x_i} e^{-2n}}{x_1!\, x_2! \ldots x_n!} \; \frac{x_1!\, x_2! \ldots x_n!}{5^{\sum_{i=1}^{n} x_i} e^{-5n}} \leq k.$$

This inequality can be written as

$\left(\frac{2}{5}\right)^{\sum_{i=1}^{n} x_i} e^{3n} \leq k$, or $(\sum_{i=1}^{n} x_i)\; In\; \left(\frac{2}{5}\right) + 3n \;\leq In\; k$.

Since In (2/5) < 0. The latter inequality is the same as

$$\sum_{i=1}^{n} x_i \;\geq\; \frac{Ink - 3n}{In\left(\frac{2}{5}\right)} = c.$$

If n = 4 and c = 13,then

$$\alpha = P\left(\sum_{i=1}^{4} x_i \;\geq 13; \lambda = 2\right) = 1 - 0.936 = 0.064.$$

Since $\sum_{i=1}^{4} x_i$ has a Poisson distribution with mean 8 when λ = 2.

Because λ is the quotient of nonnegative functions, $\lambda \geq 0$. In addition , since $\omega \subset \Omega$, it follows that $L(\widehat{\omega}) \leq L(\widehat{\Omega})$ and hence $\lambda \leq 1$. thus 0 $\leq \lambda \leq 1$. If the maximum of L in ω is much smaller than in Ω. It would seem that the data $x_1, x_2 .., x_n$ do not support the hypothesis $H_0 : \theta \in \omega$. that is . a small value of the ratio $\lambda = L(\widehat{\omega}) / L(\widehat{\Omega})$ would lead to the rejection of H_0 . In contrast, a value of the ratio λ that is close to 1 would support the null hypothesis H_0 this reasoning leads us to the next definition.

Definition: To test $H_0 : \theta \in \omega$ against $H_1 : \theta \in \omega$ the critical region for the likelihood ratio test is the set of points in the sample space for which.

$\lambda = \frac{L(\widehat{\omega})}{L(\widehat{\Omega})} \leq k,$ Where $0 < k < 1$ and k is selected so that the test has a desired significance level α. The next example illustrates these definitions

Example:

Assume that weight X in ounces of a " 10- pound " bag of sugar is N (μ .5) . We shall test the hypothesis $H_0 : \mu = 162$ against the alternative hypothesis $H_1 : \mu \neq 162$. Thus, $\Omega = \{\mu : -\infty < \mu < \infty\}$ and $\omega = \{162\}$. To find the likelihood ratio, we need $L(\widehat{\omega})$ and $L(\widehat{\Omega})$ when H_0 is true , μ can take on only one value , namely $\mu = 162$. Hence $L(\widehat{\omega}) = L(162)$. To find $L(\widehat{\Omega})$ we must find the value of μ that maximizes $L(\mu)$. Recall that $\hat{\mu} = \hat{x}$ is the maximum likelihood estimate of μ. Then $L(\widehat{\Omega}) = L(\hat{x})$ and the likelihood ratio $\lambda\; L(\widehat{\omega}) / L(\widehat{\Omega})$ is given by.

$$\lambda = \frac{(10_\mu)^{-\frac{n}{2}} exp\left[-\left(\frac{1}{10}\right)\sum_{i=1}^{n}(x_i - 162)^2\right]}{(10_\mu)^{-\frac{n}{2}} exp\left[-\left(\frac{1}{10}\right)\sum_{i=1}^{n}(x_i - \underline{x})^2\right]} = \frac{exp\left[-\left(\frac{1}{10}\right)\sum_{i=1}^{n}(x_i - \underline{x})^2 - \left(\frac{n}{10}\right)(\underline{x} - 162)^2\right]}{exp\left[-\left(\frac{1}{10}\right)\sum_{i=1}^{n}(x_i - \underline{x})^2\right]}$$

$$= exp\left[-\frac{n}{10}\left(\underline{x}-162\right)^2\right]$$

On the one hand , a value of $\underline{x}$ close to 162 would tend to support H_0, and in that case λ is close to 1 . on the other hand , an $\underline{x}$ that differs from 162 by too much would tend to support H_1. (see Figure 1 for the graph of this likelihood ratio when n=5). A critical region for likelihood ratio is given by $\lambda \leq$ k , where k is selected so that the significance level of the test is α . Using this criterion and simplifying the

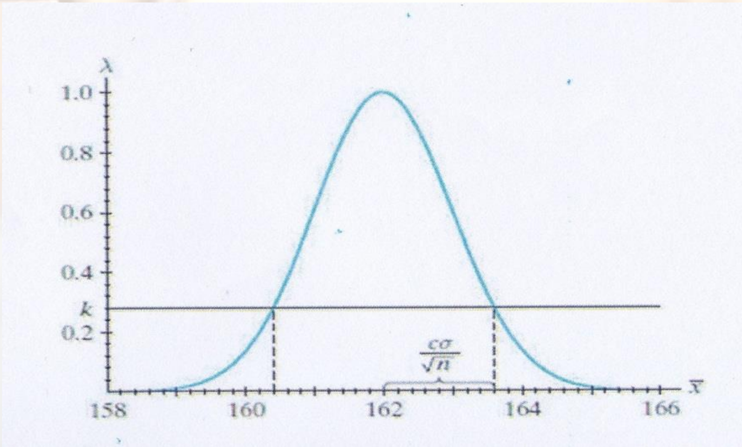

Figure 1. The likelihood ratio for testing $H_0: \mu = 162$

Inequality as we do when we use the Neyman- Pearson lemma , we find that $\lambda \leq$ k is equivalent to cach of the following Inequalities:

$$-\left(\frac{n}{10}\right)\left(\underline{x}-162\right)^2 \leq In\, k,$$

$$\left(\underline{x}-162\right)^2 \geq -\left(\frac{10}{n}\right) In\, k$$

$$\frac{\{\underline{x}-162\}}{\frac{\sqrt{5}}{\sqrt{n}}} \geq \frac{\sqrt{-\left(\frac{10}{n}\right) In\, k}}{\frac{\sqrt{5}}{\sqrt{n}}} = c$$

Since $z = \left(\underline{x}-162\right)/\left(\frac{\sqrt{5}}{\sqrt{n}}\right)$ is N(0.1) when $H_0: \mu = 162$ is true. Let $c\ = z_{\propto}/2$ Thus the critical region is .

$$c = \left\{ x \colon \frac{\left|\underline{x} - 162\right|}{\frac{\sqrt{5}}{\sqrt{n}}} \geq \ c \ = z_{\propto}/2 \right\}$$

To illustrate , if $\propto = 0.05$ then $z_{0.025} = 1.96$.