LECTURE (6)

CALCULUS 2

Certain Powers of Trigonometric
and
Hyperbolic Integrals

المصادر: CALCULUS I
CALCULUS II

Certain Powers of Trigonometric and Hyperbolic Integrals

Consider the following integrals form:

$(A)\quad \int sin^m u \; cos^n u \, du \qquad or \qquad \int sinh^m u \; cosh^n u \, du$

$(B)\quad \int tan^m u \; sec^n u \, du \qquad or \qquad \int tanh^m u \; sech^n u \, du$

$(C)\quad \int cot^m u \; csc^n u \, du \qquad or \qquad \int coth^m u \; csch^n u \, du$

Under Type *(A)* , there are three cases:

Case 1: If *m* is odd and positive integers , we factor out *sin u* *(sinh u)* and change the remaining even power of *sin u* *(sinh u)* to *cos u (cosh u)* using the identities:

$$\sin^2 u = 1 - \cos^2 u \quad , \quad \sinh^2 u = \cosh^2 u - 1$$

Example (1): Evaluate the integral $\int \sin^5 2x \ \cos^{\frac{-3}{2}} 2x \, dx$

Solution: $\int \sin^5 2x \ \cos^{\frac{-3}{2}} 2x \, dx = \int \sin^4 2x \ \cos^{\frac{-3}{2}} 2x \ \sin 2x \ dx$

$$= \int (1 - \cos^2 2x)^2 \cos^{\frac{-3}{2}} 2x \sin 2x \, dx = \int (1 - 2\cos^2 2x + \cos^4 2x) \cos^{\frac{-3}{2}} 2x \sin 2x \, dx$$

$$= \int \left(\cos^{\frac{-3}{2}} 2x - 2\cos^{\frac{1}{2}} 2x + \cos^{\frac{5}{2}} 2x\right) \sin 2x \, dx$$

$$= \int (\cos^{\frac{-3}{2}} 2x \ \sin 2x - 2\cos^{\frac{1}{2}} 2x \sin 2x + \cos^{\frac{5}{2}} 2x \sin 2x) \, dx$$

$$= \left[-\frac{1}{2} \frac{\cos^{\frac{-1}{2}} 2x}{\frac{-1}{2}} + \frac{\cos^{\frac{3}{2}} 2x}{\frac{3}{2}} + \frac{-1}{2} \frac{\cos^{\frac{7}{2}} 2x}{\frac{7}{2}} \right] + C = \cos^{\frac{-1}{2}} 2x + \frac{2}{3} \cos^{\frac{3}{2}} 2x - \frac{1}{7} \cos^{\frac{7}{2}} 2x + C$$

Case 2: If *n* is odd and positive integers , we factor out *cos u (cosh u)* and change the remaining even power of *cos u (cosh u)* to *sin u (sinh u)* using the identities:

$$\cos^2 u = 1 - \sin^2 u \qquad , \qquad \cosh^2 u = 1 + \sinh^2 u$$

Example (2): Evaluate the integral $\int sinh^4 3x\ cosh^3 3x\, dx$

Solution: $\int sinh^4 3x\ cosh^3 3x\, dx = \int sinh^4 3x\ cosh^2 3x \cosh 3x\ dx$

$$= \int sinh^4 3x\,(1 + sinh^2 3x) \cosh 3x\, dx$$

$$= \int (sinh^4 3x \cosh 3x + sinh^6 3x \cosh 3x\,)\, dx$$

$$= \left[\frac{1}{3}\frac{sinh^5 3x}{5} + \frac{1}{3}\frac{sinh^7 3x}{7}\right] + C \ = \frac{sinh^5 3x}{15} + \frac{sinh^7 3x}{21} + C$$

Case 3: If both *m* and *n* are even and positive integers (or one of them zero) , we reduce the degree of the expression by using the identities:

$$sin^2 u = \frac{1-cos2u}{2} \quad , \quad sinh^2 u = \frac{cosh2u-1}{2}$$

$$cos^2 u = \frac{1+cos2u}{2} \quad , \quad cosh^2 u = \frac{cosh2u+1}{2}$$

Example (3): Evaluate the integral $\int sin^2 2x \ cos^2 2x \, dx$

Solution: $\int sin^2 2x \ cos^2 2x \, dx = \int \left(\frac{1-cos4x}{2}\right)\left(\frac{1+cos4x}{2}\right) dx$

$$= \frac{1}{4}\int (1-\cos 4x)(1+\cos 4x)\, dx$$

$$= \frac{1}{4}\int (1-cos^2 4\,x)dx = \frac{1}{4}\int \left(1-\left(\frac{1+cos8x}{2}\right)\right) dx$$

$$= \frac{1}{4}\int \left(1-\frac{1}{2}-\frac{1}{2}\cos 8x\right) dx = \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{4}\left[\frac{x}{2}-\frac{1}{16}\sin 8x\right] + C$$

Under Type *(B)* , there are two cases:

Case 1: If *n* is even and positive integers, we factor out $sec^2u\ (sech^2u)$ and change the remaining even power of *sec u (sech u)* to *tan u (tanh u)* using the identities:

$$sec^2 u = 1 + tan^2 u \qquad , \qquad sech^2 u = 1 - tanh^2 u$$

Example (4): Evaluate the integral $\int sech^4 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx$

Solution: $\int sech^4 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx = \int sech^2 \frac{x}{2}\ sech^2 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx$

$$= \int (1 - tanh^2 \frac{x}{2})\ sech^2 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx$$

$$= \int \left(tanh^{\frac{-1}{3}} \frac{x}{2} sech^2 \frac{x}{2} - tanh^{\frac{5}{3}} \frac{x}{2} sech^2 \frac{x}{2}\right) dx = \left[2 \frac{tanh^{\frac{2}{3}} \frac{x}{2}}{\frac{2}{3}} - 2 \frac{tanh^{\frac{8}{3}} \frac{x}{2}}{\frac{8}{3}} \right] + c$$

$$= \left[tanh^{\frac{2}{3}} \frac{x}{2} - \frac{3 tanh^{\frac{8}{3}} \frac{x}{2}}{4} \right] + c$$

Case 2: If *m* is odd and positive integers, we factor out *sec u tan u (sech u tanh u)* and change the remaining even power of *tan u (tanh u)* to *sec u (sech u)* using the identities:

$$tan^2 u = sec^2 u - 1 \qquad , \qquad tanh^2 u = 1 - sech^2 u$$

Example (2): Evaluate the integral $\int tan^3 2x \; sec^{\frac{-1}{4}} 2x \, dx$

Solution: $\int tan^3 2x \; sec^{\frac{-1}{4}} 2x \, dx = \int tan^2 2x \; sec^{\frac{-5}{4}} 2x \; (\tan 2x \; \sec 2x) \, dx$

$$= \int (sec^2 2x - 1) \, sec^{\frac{-5}{4}} 2x \; (\sec 2x \tan 2x) \, dx$$

$$= \int \left[sec^{\frac{3}{4}} 2x - sec^{\frac{-5}{4}} 2x\right] (\sec 2x \tan 2x) \, dx$$

$$= \int \left[sec^{\frac{3}{4}} 2x \, (\sec 2x \tan 2x) - sec^{\frac{-5}{4}} 2x (\sec 2x \tan 2x)\right] dx$$

$$= \left[\frac{1}{2} \, \frac{sec^{\frac{7}{4}} 2x}{\frac{7}{4}} - \frac{1}{2} \, \frac{sec^{\frac{-1}{4}} 2x}{\frac{-1}{4}}\right] + C = \frac{2}{7} \, sec^{\frac{7}{4}} 2x + 2 \, sec^{\frac{-1}{4}} 2x + C$$

Under Type *(C)* , there are two cases similar to those of type *(B)* where

The identities:

Case 1: $csc^2 u = 1 + cot^2 u \quad , \quad csch^2 u = coth^2 u - 1$

Case 2: $cot^2 u = csc^2 u - 1 \quad , \quad coth^2 u = 1 + csch^2 u$

Exercises :

(1) $\int sin^5 2x \; dx$

(2) $\int \; cos^3 x \;\; cos^{\frac{-1}{2}} x \, dx.$

(3) $\int csc^6 x \; dx$.

(4) $\int tan^3 x \; \sec x \; dx.$

(5) $\int_0^1 sinh^4 x \, dx$.