المحاضرة الخامسة

Chapter (2)

The relation between field and circuit theory: Maxwell's equations.

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2.1 Introduction :

In circuit theory we deal with circuit element , the voltage (V) across them , and the total current (I) through them . In field theory we deal with the field vectors (E,D,B,H,and J)and their values as a function of position .

Consider a short rod of length (l) and cross-sectional area(A) , Thus the voltage difference between the ends of the rod is , from ohms law ,

$$V= IR \quad \ldots\ldots\ldots\ldots (1)$$

Where (I) is the current through the rod

From ohms law at point

$$E= \frac{J}{\sigma} \quad \ldots\ldots\ldots\ldots(2)$$

Where J =conduction current density, Am^{-2}

σ= conductivity, $℧m^{-1}$

Now, integrating (2) over the length of the rod, we obtain the voltage difference (V) between the ends. That is

$$V = \int E.dl = \int \frac{J}{\sigma}.dl \ldots\ldots(3)$$

$$= \frac{Jl}{\sigma} = JA\,\frac{l}{\sigma A} \ldots\ldots.(4)$$

Where JA=I = current through rod ,A

l/σA=R= resistance of rod ,Ω

A=cross section area of rod , m^2

Thus from (4) we have

$$V=IR \quad \ldots\ldots.(5)$$

Starting with field theory , we have arrived at the circuit relation known as ohm's law .

2.2 Maxwell's equation as generalizations of circuit equations

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Equation (1) it is the differential form of Maxwell's equation as derived from Ampers law

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \ldots\ldots(2)$$

Equation (2) it is the differential form of Maxwell's equation as derived from faradays law

$$\nabla \cdot D = \boldsymbol{\rho} \quad \ldots\ldots\ldots(3)$$

Equation (3) is Maxwell's electric field equation as derived from Gauss law in differential form

$$\nabla \cdot B = 0 \quad \ldots\ldots\ldots\ldots(4)$$

Equation (4) is Maxwell's magnetic field equation as derived from Gauss's law

2-3 Maxwell's equation in free space

Maxwell's equation are stated in their general form , for the special case of free space , where the current density (J) and the charge density (ρ) are zero , the equations reduce to a simple form – In integral form the equations are

$$\oint . H. dl = \int_s \frac{\partial D}{\partial t} .ds \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\oint E . dl = -\int_s \frac{\partial B}{\partial t} .ds \quad \ldots\ldots\ldots\ldots(2)$$

$$\oint_S D. ds = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\oint_s B. ds = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

-In Differential form the equations are

$$\nabla \times H = \frac{\partial D}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$\nabla .D=0$…………………(7)

$\nabla .B=0$…………………(8)

2.4 Maxwell's equation for harmonically varying fields

If we assume that the fields vary harmonically with time , Maxwell's equations can be expressed in another special form .

Thus, if (D) varies with time as given by

$D=D_0e^{j\omega t}$…………………(1)

Then $\frac{\partial D}{\partial t}=j\omega D_0e^{j\omega t}=j\omega D$……(2)

When the same assumption is made for B ,

$B=B_0e^{j\omega t}$

$\frac{\partial B}{\partial t}=j\omega B_0e^{j\omega t}=j\omega B$, $B=\mu H$

$\frac{\partial B}{\partial t}=j\omega\mu H$…………………(3)

Maxwell's equations in integral form reduce to

$$\oint H.dl=(\sigma+j\omega\varepsilon)\int_s E.ds \dots\dots\dots (4)$$

$$\oint E.dl=-j\omega\mu\int_s H.ds \dots\dots\dots\dots\dots (5)$$

$$\oint_s D.ds=\int_s \rho dV \dots\dots\dots\dots\dots\dots\dots (6)$$

$$\oint_s B.ds=0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

In differential form they are

$\nabla\times H=(\sigma+j\omega\varepsilon)E$……………(8)

$\nabla \times E = -j\omega\mu H$……………..(9)

$\nabla . D = \rho$……………........(10)

$\nabla . B = 0$ …………………(11)

2.5 Electromagnetic wave in vacuum

The wave equation for E and B

In regions of space, there is on charge or current , Maxwell's equation read

$\nabla . E = 0$

$\nabla . B = 0$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

Electromagnetic wave equation for the field vector $\vec{E}$ By applying the curl to equation(3) we get

$\nabla \times E = -\frac{\partial B}{\partial t} \rightarrow \nabla \times \nabla \times E = \nabla \times (-\frac{\partial B}{\partial t})$

$\rightarrow \nabla \times \nabla \times E = - (\nabla \times B)$

By using the identity : $\nabla \times \nabla \times E = \nabla (\nabla . E) - \nabla^2 E$.

$\nabla (\nabla . E) - \nabla^2 E = -\frac{\partial}{\partial t} (\nabla \times B) \rightarrow$ but $\nabla . E = 0$ from (1)

and $\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial B}{\partial t}$

$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$ ………………(1)

but $\varepsilon_0\mu_0 = 1/c^2$

The relation (1) becomes $\nabla^2 E = 1/c^2 \frac{\partial^2 y}{\partial t^2}$ ....(2)

Electromagnetic wave equation for the field vector $\vec{B}$ By applying the curl to equation (4) we get

$$\nabla \times B = \mu_0\varepsilon_0 \frac{\partial E}{\partial t} \longrightarrow \nabla \times \nabla \times B = \nabla \times (\mu_0\varepsilon_0) \frac{\partial E}{\partial t}$$

$$\longrightarrow \nabla \times \nabla \times B = \mu_0\varepsilon_0 \frac{\partial}{\partial t} (\nabla \times E)$$

By using the identity $\nabla \times \nabla \times B = \nabla (\nabla . B) - \nabla^2 B$.

$\nabla (\nabla . B) - \nabla^2 B = \mu_0\varepsilon_0 \frac{\partial}{\partial t} (\nabla \times E) \longrightarrow$ but $\nabla . B = 0$ from (2)

and $\nabla \times E = -\frac{\partial B}{\partial t}$ from (3)

$-\nabla^2 B = -\mu_0\varepsilon_0 \frac{\partial}{\partial t} (\frac{\partial B}{\partial t})$ $\nabla^2 B = \mu_0\varepsilon_0 \frac{\partial^2 B}{\partial t^2}$ ....(3)

but $\varepsilon_0\mu_0 = \frac{1}{c^2}$

the relation between the electric $\vec{E}$ and the magnetic vector $\vec{B}$ is

E=CB ....(5)

Where (C) is the speed of light or the speed of electromagnetic wave and equal $C = 3\times10^8$ mls .

EX / A parallel – plate capacitor of radius (R) and separation (d) has a voltage at the center given
by $V = V_0$ sinwt . as function of radius (r) (for $r < R$) find (a) the displacement density $J_d(r)$

(b) the magnetic field H (r) , take $d << R$

Sol / $\oint_c H.dl = \int_s (J_c + J_d).ds$

$\oint_c H.dl = \int_s (\sigma E + \frac{\partial D}{\partial t})$ .ds $\longrightarrow$ $J_c = \sigma E = 0$ $\longrightarrow$ $\sigma = 0$

a/ $J_d = \int_s (\varepsilon_0 \frac{\partial E}{\partial t}).ds$

$J_d = \varepsilon_0 \frac{d}{dt}(\frac{v_0 \sin \omega t}{d})$ $\longrightarrow$ $J_d = \frac{\varepsilon_0 \ \omega v_0 \ \cos \omega t}{d}$

b/ $\oint_c H.dl = \int_s J_d .ds$

$H.2\pi r = \frac{\varepsilon_0 \omega v_0 \ cos\, \omega t}{d} .\pi r^2$

$$H = \frac{\varepsilon_0 \ \omega v_0 \cos \omega t .\pi r^2}{2d.\pi r}$$

$$H = \frac{r\varepsilon_0 \ v_0 \cos \omega t}{2d}$$