Chapter Two

Basic Algebra, Geometry and Trigonometry formulas

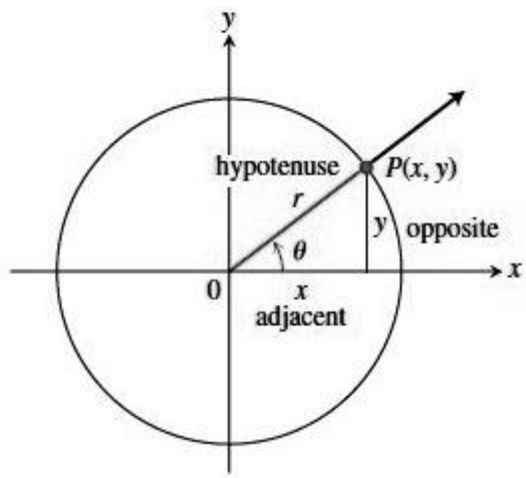
Trigonometry formulas

Definitions and fundamental identities

Sine $\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$

Cosine $\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$

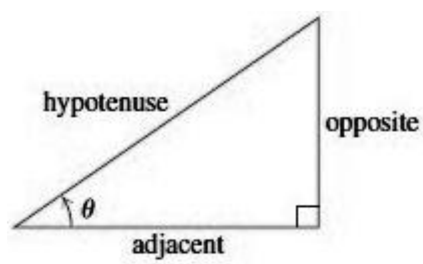
Tangent $\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$



$\sin\theta = \frac{\text{opp}}{\text{hyp}} =$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} =$ $\sec\theta = \frac{\text{hyp}}{\text{adj}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} =$ $\cot\theta = \frac{\text{adj}}{\text{opp}}$



Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

The Basic Trigonometry function

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} =$ $\qquad \cot\theta = \dfrac{1}{\tan\theta}$ $\qquad \sec\theta = \dfrac{1}{\cos\theta} =$ $\qquad \csc\theta = \dfrac{1}{\sin\theta}$

Even	odd
$\cos(-x) = \cos(x)$	$\sin(-x) = -\sin(x)$
$\sec(-x) = \sec(x)$	$\tan(-x) = -\tan(x)$
	$\csc(-x) = -\csc(x)$
	$\cot(-x) = -\cot(x)$

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \qquad (1)$$

This equation, true for all values of θ is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$\boxed{\begin{aligned} 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}}$$

Addition formulas

$$\boxed{\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \sin(A+B) &= \sin A\cos B + \cos A\sin B \end{aligned}} \qquad (2)$$

Double –Angle Formulas

$$\boxed{\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned}} \qquad (3)$$

Additional formulas come from combining the equations $\sin^2\theta + \cos^2\theta = 1$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}} \qquad (4)$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}} \qquad (5)$$

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x+p)$ for every value of x. The smallest such value of p is the period of f.

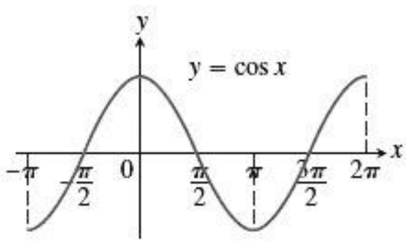
Periodic Trigonometric function

Periodic π

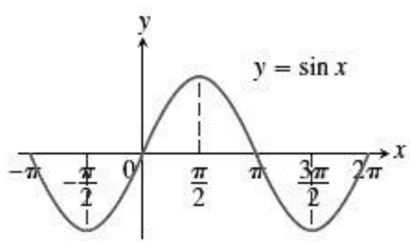
$$\tan(x+\pi) = \tan x$$
$$\cot(x+\pi) = \cot x$$

Periodic 2π

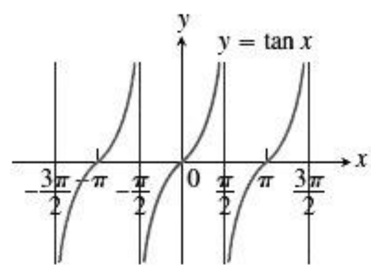
$$\sin(x+2\pi) = \sin x$$
$$\cos(x+2\pi) = \cos x$$
$$\sec(x+2\pi) = \sec x$$
$$\csc(x+2\pi) = \csc x$$



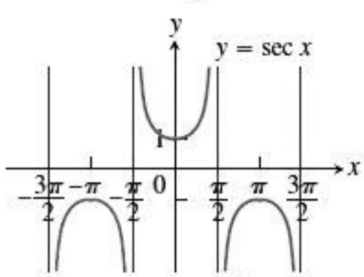
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(a)



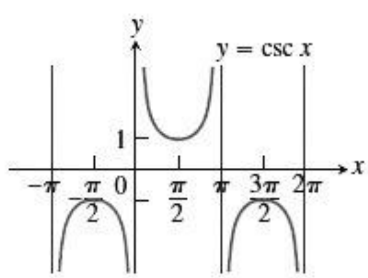
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(b)



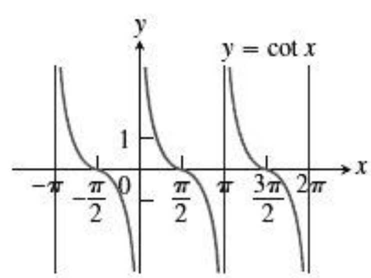
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π
(c)



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π
(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π
(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π
(f)

Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Identities

$\sin(-\theta) = -\sin\theta$ $\qquad \cos(-\theta) = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$ $\qquad \sec^2\theta = 1 + \tan^2\theta$ $\qquad \csc^2\theta = 1 + \cot^2\theta$

$\sin 2\theta = 2\sin\theta\,\cos\theta$ $\qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ $\qquad \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\qquad \tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin(A - \frac{\pi}{2}) = -\cos A$ $\qquad \cos(A - \frac{\pi}{2}) = \sin A$

$\sin(A + \frac{\pi}{2}) = \cos A$ $\qquad \cos(A + \frac{\pi}{2}) = -\sin A$

$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$

$\sin A\cos B = \frac{1}{2}\sin(A-B) - \frac{1}{2}\sin(A+B)$

$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

Algebra

Arithmetic operations

$$a(b+c)=ab+ac \qquad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd} \qquad \frac{a}{b}+\cdot\frac{c}{d}=\frac{ad+bc}{bd} \qquad \frac{a/b}{c/d}=\frac{a}{b}\cdot\frac{d}{c}$$

Law of sings

$$-a(-a)=a \qquad \frac{-a}{b}=-\frac{a}{b}=\frac{a}{-b}$$

Zero Division by zero not defined

If $a \neq 0$: $\quad \frac{0}{a}=0, \quad a^0=1 \quad 0^a=0$

For any number a: $\quad 0\cdot a=a\cdot 0=0$

Law of exponents

$$a^m a^n=a^{m+n} \qquad (ab)^m=a^m b^m \qquad (a^m)^n=a^{mn} \qquad a^{m/n}=\sqrt[n]{a^m}=(\sqrt[n]{a})^m$$

If $a \neq 0$,

$$\frac{a^m}{a^n}=a^{m-n}\ , \quad a^0=1 \qquad a^{-m}=\frac{1}{a^m}$$

The Binomial Theorem for any positive integer n

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\cdots nab^{n-1}+b^n$$

For instant

$$(a+b)^2=a^2+2ab+b^2 \qquad (a-b)^2=a^2-2ab+b^2$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3 \qquad (a-b)^3=a^3-3a^2b+3ab^2-b^3$$

Factoring the Deference of Like Integer Power $n>1$

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots\cdots+ab^{n-2}+b^{n-1})$$

For instant

$$a^2-b^2=(a-b)(a+b)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$a^4-b^4=(a-b)(a^3+a^2b+ab^2+b^3)$$

Completing the Square

If $a\neq 0$

$$ax^2+bx+c=a(x^2+\frac{b}{a}x)+c$$

$$=a(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2})+c$$

$$=a(x^2+\frac{b}{a}x+\frac{b^2}{4a^2})+a(-\frac{b^2}{4a^2})+c$$

$$=a(x^2+\frac{b}{a}x+\frac{b^2}{4a^2})+c-\frac{b^2}{4a}$$

This is $=(x+\frac{b}{2a})^2$ this is part C

$$=au^2+C \qquad (u=x+(b/2a))$$

The Quadratic Formula

If $a\neq 0$ and $ax^2+bx+c=0$ then

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

EXAMPLE 1: Prove the following identities $\sec^2 x+\csc^2 x=\sec^2 x\csc^2 x$

Sol

L.H.S $\sec^2 x+\csc^2 x=$

$$\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}=\frac{\sin^2 x+\cos^2 x}{\cos^2 x\sin^2 x}$$

$$\sin^2 x+\cos^2 x=1 \quad\Rightarrow\quad =\frac{1}{\cos^2 x\sin^2 x}$$

$$=\frac{1}{\cos^2 x}\cdot\frac{1}{\sin^2 x}=\sec^2 x\cdot\csc^2 x$$

R.H.S

EXAMPLE 2: Prove that $\csc\theta + \tan\theta\sec\theta = \csc\theta\sec^2\theta$

Sol

L.H.S $\csc\theta + \tan\theta\sec\theta = \dfrac{1}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\cos\theta}$

$$= \frac{1}{\sin\theta} + \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin\theta\sin\theta}{\sin\theta\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos^2\theta} = \frac{1}{\sin\theta\ \cos^2\theta}$$

$$= \frac{1}{\sin\theta}\cdot\frac{1}{\cos^2\theta} = \csc\theta\ \sec^2\theta$$

R.H.S

EXAMPLE 3: Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

Sol

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Prove that $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$ (H.W)

EXAMPLE 4: Prove the following identities $\cos^4\theta-\sin^4\theta=\cos^2\theta-\sin^2\theta$

Sol

L.H.S $\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)$

$$=(\cos^2\theta-\sin^2\theta)(1)$$

$$=(\cos^2\theta-\sin^2\theta)\quad \text{R.H.S}$$

EXAMPLE 5: Prove that $\sin 2\theta=2\sin\theta\cos\theta$

Sol:

$$\sin 2\theta=\sin(\theta+\theta)=\sin\theta\cos\theta+\cos\theta\sin\theta$$

$$=2\sin\theta\cos\theta$$

EXAMPLE 6: Prove that $\cos 2\theta=\cos^2\theta-\sin^2\theta$

Sol:

$$\cos 2\theta=\cos(\theta+\theta)=\cos\theta\cos\theta-\sin\theta\sin\theta$$

$$=\cos^2\theta-\sin^2\theta$$

$$=2\cos^2\theta-1$$

$$=1-2\sin^2\theta$$

EXAMPLE 7: Prove that $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

Sol:

$$2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)=2[\sin\tfrac{A}{2}\cos\tfrac{B}{2}+\cos\tfrac{A}{2}\sin\tfrac{B}{2}]\cdot[\cos\tfrac{A}{2}\cos\tfrac{B}{2}+\sin\tfrac{A}{2}\sin\tfrac{B}{2}]$$

$$=2[\sin\tfrac{A}{2}\cos\tfrac{B}{2}\cdot\cos\tfrac{A}{2}\cos\tfrac{B}{2}+\cos\tfrac{A}{2}\sin\tfrac{B}{2}\cdot\cos\tfrac{A}{2}\cos\tfrac{B}{2}$$

$$+\sin\tfrac{A}{2}\cos\tfrac{B}{2}\cdot\sin\tfrac{A}{2}\sin\tfrac{B}{2}+\cos\tfrac{A}{2}\sin\tfrac{B}{2}\cdot\sin\tfrac{A}{2}\sin\tfrac{B}{2}]$$

$$=2[\sin\tfrac{A}{2}\cos^2\tfrac{B}{2}\cdot\cos\tfrac{A}{2}+\cos^2\tfrac{A}{2}\sin\tfrac{B}{2}\cdot\cos\tfrac{B}{2}+\sin^2\tfrac{A}{2}\cos\tfrac{B}{2}\cdot\sin\tfrac{B}{2}+\cos\tfrac{A}{2}\sin^2\tfrac{B}{2}\cdot\sin\tfrac{A}{2}]$$

$$=2[(\sin\tfrac{A}{2}\cos\tfrac{A}{2})(\cos^2\tfrac{B}{2}+\sin^2\tfrac{B}{2})+(\cos\tfrac{B}{2}\sin\tfrac{B}{2})(\sin^2\tfrac{A}{2}+\cos^2\tfrac{A}{2}]$$

$= 2[(\sin\tfrac{A}{2}\cos\tfrac{A}{2}) + (\cos\tfrac{B}{2}\sin\tfrac{B}{2})]$

$= 2\sin\tfrac{A}{2}\cos\tfrac{A}{2} + 2\cos\tfrac{B}{2}\sin\tfrac{B}{2}$

$= \sin A + \sin B$

H.W **Ex 8:** Prove that

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

EXAMPLE 9: Use the addition formulas to derive the identity

1. $\cos(x - \pi/2) = \sin x$
2. $\sin(x + \pi/2) = \cos x$
3. $\cos(x + \pi/2) = -\sin x$
4. $\sin(x - \pi/2) = -\cos x$

1. $\cos(x - \pi/2) = \cos(x + (-\pi/2)) = \cos x\cos(-\pi/2) - \sin x\sin(-\pi/2)$

   $= \cos x(0) - \sin x(-1) = \sin x$

2. $\cos(x + \pi/2) = \cos x\cos(\pi/2) - \sin x\sin(\pi/2)$

   $= \cos x(0) - \sin x\,(1) = -\sin x$

3. $\sin(x + \pi/2) = \sin x\cos\pi/2 + \cos x\sin\pi/2$
   $= \sin x(0) + \cos x\,(1) = \cos x$

4. $\sin(x - \pi/2) = \sin(x + (-\pi/2)) = \sin x\cos(-\pi/2) + \cos x\sin(-\pi/2)$
   $= \sin x(0) + \cos x\,(-1) = -\cos x$

EXAMPLE 10: Solve the following equation $\tan\theta = 2\sin\theta$

Sol:

$$\tan\theta = 2\sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$

$$\sin\theta = 2\sin\theta\ \cos\theta$$

$$\sin\theta - 2\sin\theta\ \cos\theta = 0$$

$$\sin\theta(1-2\cos\theta) = 0$$

$$\sin\theta = 0 \quad \Rightarrow \quad \theta = \sin^{-1}0 = 0$$

OR $\quad 1-2\cos\theta = 0$

$$2\cos\theta = 1$$

$$\cos\theta = \tfrac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\tfrac{1}{2}$$

$$\theta = 60$$

EXAMPLE 11: Solve the following equation $1+\cos\theta = 2\sin^2\theta$

Sol:

$$1+\cos\theta = 2\sin^2\theta$$

$$1+\cos\theta - 2\sin^2\theta = 0$$

$$1+\cos\theta - 2(1-\cos^2\theta) = 0$$

$$1+\cos\theta - 2 + 2\cos^2\theta = 0$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$\cos\theta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\cos\theta = \frac{-1 \pm \sqrt{1-4(2)(-1)}}{2(2)}$$

$$\cos\theta = \frac{-1 \pm 3}{4}$$

$$\cos\theta = \frac{-1-3}{4} \quad \Rightarrow \quad \cos\theta = -1$$

$$\theta = \cos^{-1}(-1) = \pi$$

$$\cos\theta = \frac{-1+3}{4} \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$$

$\theta = \cos^{-1}(1/2) = 60$

EXAMPLE 12: Solve the following equation $\tan^2\theta + \tan\theta = 0$

Sol:

$\tan^2\theta + \tan\theta = 0$

$\tan\theta\,(\tan\theta + 1) = 0$

either $\tan\theta = 0 \quad \Rightarrow \quad \theta = -180, 0, 180$

or $\tan\theta + 1 = 0 \quad \Rightarrow \quad \tan\theta = -1$

$\theta = -45, 135$

EXAMPLE 13: Solve the following equation $\cot\theta = 5\cos\theta$

Sol:

$\frac{\cos\theta}{\sin\theta} = 5\cos\theta \quad \Rightarrow \quad 5\cos\theta\sin\theta - \cos\theta = 0$

$\cos\theta(5\sin\theta - 1) = 0$

either $\cos\theta = 0 \quad \Rightarrow \quad \theta = -\frac{\pi}{2}, \frac{\pi}{2}$

or $5\sin\theta - 1 = 0 \quad \Rightarrow \quad 5\sin\theta = 1 \quad \sin\theta = \frac{1}{5}$

$\theta = \sin^{-1}\frac{1}{5} = 11.536$

$180 - 11.536$

$\theta = 168.47$

$\theta = 90, -90, 11.536, 168.47$

EXAMPLE 14: Solve the following equation $3\cos\theta + 2\sec\theta + 7 = 0$

Sol:

$3\cos\theta + 2\frac{1}{\cos\theta} + 7 = 0$

$$3\cos^2\theta + 2 + 7\cos\theta = 0$$
$$(3\cos\theta + 1)(\cos\theta + 2) = 0$$

either $(3\cos\theta + 1) = 0 \Rightarrow \cos\theta = -1/3 \qquad \theta = 109.26$

$\because (\cos(-\theta)) = (\cos(\theta))$ even

$\theta = -109.26$

or $(\cos\theta + 2) = 0$

$\cos\theta = -2$ يهمل $\quad -1 \le \cos\theta \le 1$

$\theta = 109.26, -109.26$

EXAMPLE 15: Solve the following equation $3\tan\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Sol:

$$3\tan\theta - 3\tan^3\theta = 2\tan\theta$$
$$3\tan\theta - 3\tan^3\theta - 2\tan\theta = 0$$
$$\tan\theta - 3\tan^3\theta = 0$$
$$\tan\theta(1 - 3\tan^2\theta) = 0$$

either $\tan\theta = 0 \Rightarrow \theta = 0, 180, 360$

or $1 - 3\tan^2\theta = 0 \Rightarrow -3\tan^2\theta = -1$

$\tan^2\theta = 1/3 \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{3}}$

$\tan\theta = -\frac{1}{\sqrt{3}} \quad \theta = 150, 330$

$\tan\theta = \frac{1}{\sqrt{3}} \quad \theta = 30,\ 210$

$\theta = \{0, 30, 150, 180, 210, 360\}$

EXAMPLE 16: Solve the following equation $\sin 2\theta\cos\theta + \sin^2\theta = 1$

Sol:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$2\sin\theta\cos\theta\cos\theta + \sin^2\theta = 1$$

$$2\sin\theta\cos^2\theta+\sin^2\theta=1$$

$$2\sin\theta\cos^2\theta+(1-\cos^2\theta)=1$$

$$2\sin\theta\cos^2\theta+1-\cos^2\theta=1$$

$$2\sin\theta\cos^2\theta+1-1-\cos^2\theta=0$$

$$2\sin\theta\cos^2\theta-\cos^2\theta=0$$

$$\cos^2\theta(2\sin\theta-1)=0$$

$$\cos^2\theta=0 \quad \Rightarrow \quad \theta=90,\ 270$$

$$2\sin\theta-1=0 \quad \Rightarrow \quad \sin\theta=1/2 \quad \Rightarrow\theta=30,150$$

$$\theta=\{30,\ 90,150,\ 270\}$$

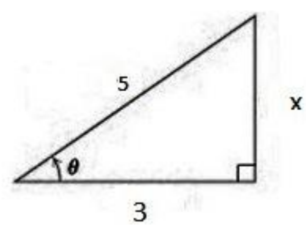
EXAMPLE 17: If $\cos\theta=3/5$ find $\sin\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, $\cot\theta$

Sol

$\sin\theta=4/5$ $\qquad$ $\sec\theta=5/3$

$\tan\theta=4/3$ $\qquad$ $\cot\theta=3/4$

$\csc\theta=5/4$



$x^2+3^2=5^2$

$x^2=25-9$

$x^2=16$

$x=4$

EXAMPLE 18: If $\tan A=-1/7$, $\tan B=3/4$ find $\tan(A-B)$

Sol

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

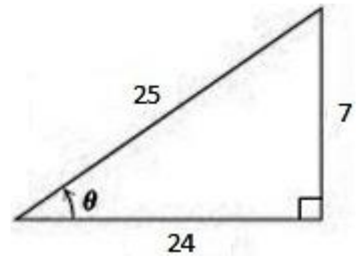
$$=\frac{-1/7-3/4}{1+(-1/7)(3/4)}=-1$$

$$\tan(A-B)=-1$$

$$(A-B)=135$$

EXAMPLE 19: If $\tan\theta = 7/24$ find without using calculator. Find 1. $\sec\theta$, 2. $\sin\theta$

Sol:



$$\tan\theta = 7/24$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{24/25} = \frac{25}{24}$$

$$\sin\theta = 7/25$$

EXAMPLE 1: Sketch $y = 3\sin 2x$

Sol:

$2x = 0 \quad \therefore x = 0$

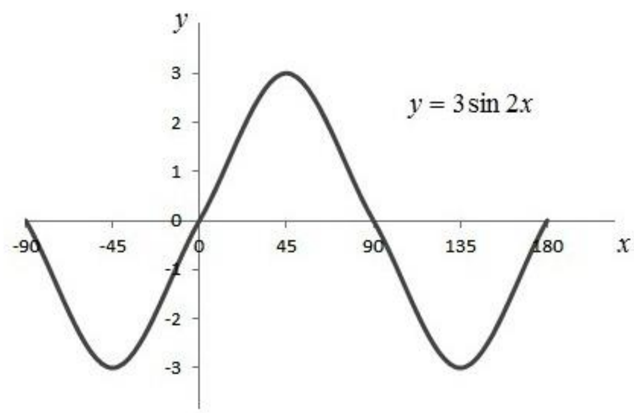
$2x = \pi/2 \quad \therefore x = \pi/4$

$2x = \pi \quad \therefore x = \pi/2$

$2x = 3\pi/2 \quad \therefore x = 3\pi/4$

$2x = 2\pi \quad \therefore x = \pi$

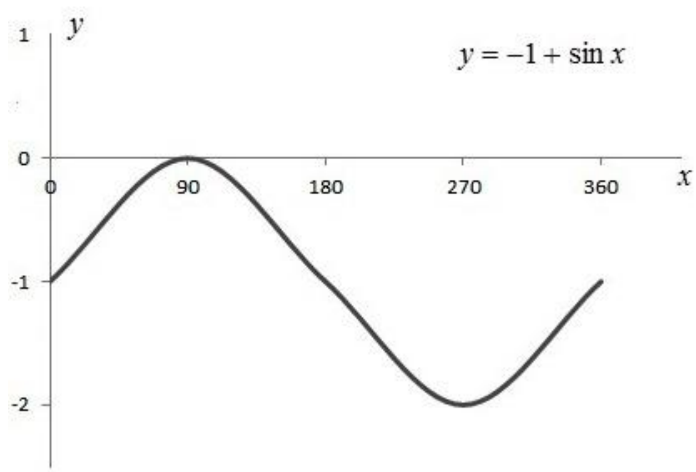
x	y
0	0
$\pi/4$	3
$\pi/2$	0
$3\pi/4$	-3
π	0



EXAMPLE 2: Sketch $y = -1 + \sin x$

Sol:

x	y
0	-1
$\pi/2$	0
π	-1
$3\pi/2$	-2
2π	-1



H.W **Ex 3:** Draw

1. $y = 1 + \sin x$
2. $y = (1/6)\cos x$
3. $y = \cos x + 2$
4. $y = \sin^2 x$
5. $y = 5 + \sin^2 x$
6. $y = -3 + \cos^2 x$

Transcendental function

1.1 Logarithm function

- **Definition** $\log_a x$
- **Properties of Logarithm function**
- **Rule of Logarithm function**
- **Example**

1.2 Exponential function

- **Definition of**
- **Properties and rule of Exponential function**
- **Example**

1.3 Invers function

- **Example**

1.1 Logarithm function

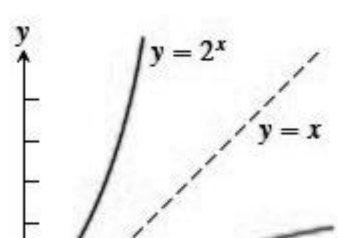
Logarithms with Base a

Definition $\log_a x$

For any positive number $a \neq 1$ $\log_a x$ is the inverse function of a^x .

Example: $y = \log_2 x$ reflecting the graph of $y = a^x$ when $a = 2$ as shown in Fig.

So that mean



$y = \log_a x$

$a^y = x$

Inverse Equations for a^x and $\log_a x$

1. $a^{\log_a x} = x$ $\quad\quad x > 0$
2. $\log_a(a^x) = x$ $\quad\quad$ all x

Rules for base a logarithms for any numbers $x > 0$ and $y > 0$

1. Product Rule: $\log_a xy = \log_a x + \log_a y$

2. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

3. Reciprocal Rule: $\log_a \frac{1}{y} = -\log_a y$

4. Power Rule: $\log_a x^y = y \log_a x$

Also

1. $\log_a a = 1$
2. $\log_a 1 = 0$
3. $\log_a x = \frac{\ln x}{\ln a}$
4. $\log_a x = \frac{\log x}{\log a}$
5. $\log x = \frac{\ln x}{\ln 10}$

EXAMPLE 1: Prove that $\log_a x = \dfrac{\ln x}{\ln a}$

Proof: $a^{\log_a x} = x$

tak ln $\Rightarrow \ln a^{\log_a x} = \ln x$ using properties

$\log_a x \cdot \ln a = \ln x$

$$\log_a x = \frac{\ln x}{\ln a}$$

EXAMPLE 2: Calculate $\dfrac{1}{\log_{10} 30} + \dfrac{1}{\log_3 30}$

Sol:

$$\frac{1}{\frac{\log 30}{\log 10}} + \frac{1}{\frac{\log 30}{\log 3}} \quad or \quad \frac{1}{\frac{\ln 30}{\ln 10}} + \frac{1}{\frac{\ln 30}{\ln 3}}$$

$$\Rightarrow \frac{\ln 10}{\ln 30} + \frac{\ln 3}{\ln 30} = \frac{\ln 10 + \ln 3}{\ln 30}$$

$$= \frac{\ln(10 \times 3)}{\ln 30} = \frac{\ln 30}{\ln 30} = 1$$

EXAMPLE 3: Evaluate

1. $2^{\log_2(3)}$

Sol: $a^{\log_a x} = x$

$2^{\log_2(3)} = 3$

2. $\log_{10}(10)^{-7}$

Sol:

$\log_{10}(10)^{-7} \;\Rightarrow\; \log_a a^x = x$

$\log_{10}(10)^{-7} = -7$

3. $\log_2(1/4) = -2$ (H.W)
4. $\log_a x^n = \dfrac{\ln x^n}{\ln a} = n\dfrac{\ln x}{\ln a} = n\log_a x$

EXAMPLE 4: If $y = \log_3 x$ find the value of x

Sol:

$y = \log_a x \quad \Rightarrow a^y = x$

$3^y = x \quad \Rightarrow\; x = 3^y$

EXAMPLE 5: calculate $y = \log_4 8$

Sol:

$x = a^y \qquad 4^y = 8$

$2^{2y} = 2^3$

$2y = 3 \;\Rightarrow\; y = 3/2$

H.W **Ex 6:** Calculate

1. $y = \log_2 8$
2. $y = \log_7 49$

EXAMPLE 7: Find the value of x

1. $\log_3 x = 4$

   Sol:

   $a^y = x \quad \Rightarrow\; 3^4 = x$

   $x = 81$
2. $\log_{64} x = 6$ (H.W)
3. $\log_5(1/125) = x$ (H.W)

H.W **Ex 8:** Find value of y

1. $y = 5^{\log_5 7} \quad \Rightarrow\; y = 7$ 2. $y = \log_6 36 \quad \Rightarrow\; y = 2$

3. $y = \log_3(1/9) \quad \Rightarrow \quad y = -2$ 4. $y = \log_3 \sqrt{3} \quad \Rightarrow \quad y = 1/2$

5. $y = \log_4(4^{2/3}) \quad \Rightarrow \quad y = 2/3$ 6. $y = \log_x(1/\sqrt{x}) \quad \Rightarrow \quad y = -1/2$

Exponential function

$$y = a^x \quad \Rightarrow a: \text{constant}$$

$$y = e^x \quad \Rightarrow \quad e \approx 2.718$$

Properties and rule of Exponential function

1. $e^x \cdot e^y = e^{x+y}$
2. $e^x / e^y = e^{x-y}$
3. $(e^x)^n = e^{nx}$

Rule

1. $\ln e = 1$
2. $\ln 1 = 0$
3. $\ln e^u = u \ln e = u$
4. $e^{\ln 1} = e^0 = 1$
5. $e^{\ln u} = u$

EXAMPLE 1: Solve for $e^{\ln y + 2x} = e^{\ln(x+1)}$
Sol:

$$e^{\ln y + 2x} = e^{\ln(x+1)}$$

$$e^{\ln y} \cdot e^{2x} = (x+1)$$

$$y \cdot e^{2x} = (x+1)$$

$$y = (x+1)/e^{2x}$$

EXAMPLE 2: Solve for $\ln y - \ln x = \ln(x^2 + 1)$
Sol:

$$\ln y / x = \ln(x^2 + 1)$$

$$e^{\ln y/x} = e^{\ln(x^2+1)}$$
$$y/x = x^2 + 1$$
$$y = x^3 + x$$

EXAMPLE 3: $e^{y^2+1} = x + 2$

Sol:

$$e^{y^2+1} = x + 2$$
$$\ln e^{y^2+1} = \ln(x+2)$$
$$y^2 + 1 = \ln(x+2)$$
$$y^2 = \ln(x+2) - 1$$
$$y = \pm\sqrt{\ln(x+2) - 1}$$

1.2 Invers function

IF $y = \sin^{-1} x \Rightarrow x = \sin y$

OR $x = \sin y \Rightarrow y = \sin^{-1} x$

EXAMPLE 1: Prove that $\sin^{-1}(-x) = -\sin^{-1} x$

Sol:

Let $y = \sin^{-1}(-x)$
$$-x = \sin y$$
$$x = -\sin y$$
$$y = -\sin^{-1}(x)$$

EXAMPLE 2: Prove that $\sec^{-1} x = \cos^{-1}(1/x)$

Sol:

Let $y = \sec^{-1} x$ L.H.S
$$x = \sec y$$
$$x = 1/\cos y$$
$$1/x = \cos y$$
$$y = \cos^{-1}(1/x)$$

So $\sec^{-1} x = \cos^{-1}(1/x)$ R.H.S

EXAMPLE 3: Prove that $\tan^{-1} x = \cot^{-1}(1/x)$

Sol:

Let $y = \tan^{-1} x$

$x = \tan y$

$x = 1/\cot y$

$\cot y = 1/x$

$y = \cot^{-1}(1/x)$

$y = \tan^{-1} x = \cot^{-1}(1/x)$

EXAMPLE 4: Prove that $\sin^{-1} x + \cos^{-1} x = \pi/2$

Sol:

$y = \sin^{-1} x$

$x = \sin y$

$x = \cos(\pi/2 - y)$ $\quad\quad$ $\sin\theta = \cos(90 - \theta)$

$(\pi/2 - y) = \cos^{-1} x$

$(\pi/2 - \sin^{-1} x) = \cos^{-1} x$

$\pi/2 = \cos^{-1} x + \sin^{-1} x$

EXAMPLE 5: Prove that $\cos^{-1} x + \cos^{-1}(-x) = \pi$

Sol:

$y = \cos^{-1}(-x)$

$-x = \cos y$

$x = -\cos y$ $\quad\quad$ $\cos(\pi - y) = -\cos y$

$x = \cos(\pi - y)$

$(\pi - y) = \cos^{-1}(x)$

$\pi - \cos^{-1}(-x) = \cos^{-1}(x)$

$\pi = \cos^{-1} x + \cos^{-1}(-x)$

EXAMPLE 6: Prove that $\cot^{-1} x = -\tan^{-1} x + \pi/2$

Sol:

Let $y = \cot^{-1} x$

$x = \cot y$ $\qquad\qquad$ $\tan(\pi/2 - y) = \cot y$

$x = \tan(\pi/2 - y)$

$(\pi/2 - y) = \tan^{-1} x$

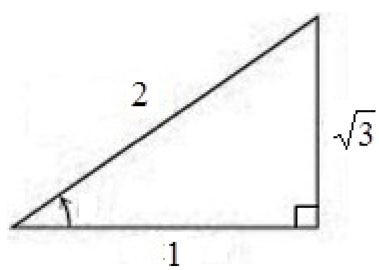
$(\pi/2 - \cot^{-1} x) = \tan^{-1} x$

$\cot^{-1} x = -\tan^{-1} x + \pi/2$

EXAMPLE 7: If $\alpha = \sin^{-1}\sqrt{3}/2$ find $\cos\alpha$, $\sec\alpha$, $\csc\alpha$, $\tan\alpha$, $\cot\alpha$

Sol:

1. $\cos\alpha = 1/2$
2. $\sec\alpha = 2$
3. $\csc\alpha = 2/\sqrt{3}$
4. $\tan\alpha = \sqrt{3}$
5. $\cot\alpha = 1/\sqrt{3}$



$4 = (\sqrt{3})^2 + x^2$

$x = 1$

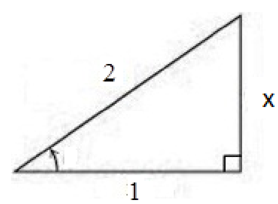
EXAMPLE 8: Evaluation the following equation

1) $\sec(\cos^{-1} 1/2)$

$y = \cos^{-1} 1/2$

$\cos y = 1/2$

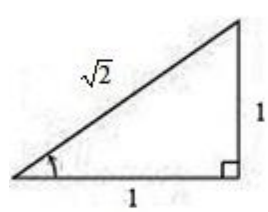
$\sec y = 2/1$



2) $\cos^{-1}(-\sin^{-1} \pi/2) = \cos^{-1}(-1) = \pi$

3) $\sin(\cos^{-1} 1/\sqrt{2})$

$\cos y = 1/\sqrt{2}$



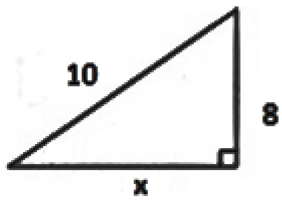
$$\sin y = 1/\sqrt{2}$$
$$\sin(\cos^{-1} 1/\sqrt{2}) = 1/\sqrt{2}$$

4. $\cos(\sin^{-1} 0.8)$

$y = \sin^{-1} 0.8$

$\sin y = 0.8$

$\cos(\sin^{-1} 0.8) = 0.6$



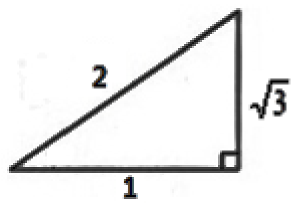
$10^2 = 8^2 + x^2$

$x = 6$

5. $\csc(\sec^{-1} 2)$

$y = \sec^{-1} 2 \quad \Rightarrow \sec y = 2$

$\csc(\sec^{-1} 2) = \csc(y) = 2/\sqrt{3}$



Hyperbolic Function

<u>Definition of hyperbolic function</u>

1. $\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{(e^x - e^{-x})}{2}$

2. $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{(e^x + e^{-x})}{2}$

3. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$

4. $\coth x = \frac{\cosh x}{\sinh x} = \frac{(e^x + e^{-x})}{(e^x - e^{-x})}$

5. $\text{sechx} = \frac{1}{\cosh x} = \frac{2}{(e^x + e^{-x})}$

6. $\operatorname{cschx} = \dfrac{1}{\sinh x} = \dfrac{2}{(e^{x} - e^{-x})}$

Identities

1. $\cosh^2 x - \sinh^2 x = 1$ Prove?
2. $\sinh 2x = 2\sinh x \cosh x$ Prove?
3. $\cosh 2x = \cosh^2 x + \sinh^2 x$ Prove?
4. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$ Prove?
5. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$ Prove?
6. $\tanh^2 x = 1 - \operatorname{sech}^2 x$ Prove?
7. $\cosh^2 x = 1 + \operatorname{csch}^2 x$ Prove?
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
10. $\sinh x + \cosh x = e^{x}$
11. $\cosh 2x = 2\sinh^2 x + 1$
12. $\cosh 2x = 2\cosh^2 x - 1$
13. $\cosh(-x) = \cosh x$
14. $\sinh(-x) = -\sinh x$
15. $\cosh x - \sinh x = e^{-x}$

EXAMPLE 1: Prove that $\sinh x + \cosh x = e^{x}$

Sol:

$$\sinh x = \frac{(e^{x} - e^{-x})}{2} \qquad \cosh x = \frac{(e^{x} + e^{-x})}{2}$$

$$\text{L.H.S } \frac{(e^{x} - e^{-x})}{2} + \frac{(e^{x} + e^{-x})}{2} = \frac{e^{x}}{2} - \frac{e^{-x}}{2} + \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

$$=\frac{e^{x}}{2}+\frac{e^{x}}{2}=e^{x}\ \text{R.H.S}$$

EXAMPLE 2: Prove that $\cosh x-\sinh x=e^{-x}$

Sol

$$\text{L.H.S}\ \frac{(e^{x}+e^{-x})}{2}-\frac{(e^{x}-e^{-x})}{2}=\frac{e^{x}}{2}+\frac{e^{-x}}{2}-\frac{e^{x}}{2}+\frac{e^{-x}}{2}$$

$$=\frac{e^{-x}}{2}+\frac{e^{-x}}{2}=e^{-x}\ \text{R.H.S}$$

EXAMPLE 3: Prove that $\cosh^{2}x-\sinh^{2}x=1$

Sol

$$\text{L.H.S}\ =\left(\frac{(e^{x}+e^{-x})}{2}\right)^{2}-\left(\frac{(e^{x}-e^{-x})}{2}\right)^{2}$$

$$=(\frac{1}{4}e^{2x}+\frac{2e^{x}e^{-x}}{4}+\frac{1}{4}e^{-2x})-(\frac{1}{4}e^{2x}-\frac{2e^{x}e^{-x}}{4}+\frac{1}{4}e^{-2x})$$

$$=\frac{e^{2x}}{4}+\frac{1}{2}+\frac{e^{-2x}}{4}-\frac{e^{2x}}{4}+\frac{1}{2}-\frac{e^{-2x}}{4}$$

$$=\frac{1}{2}+\frac{1}{2}=1\ \text{R.H.S}$$

EXAMPLE 4: Prove that $\tanh^{2}x+\text{sech}^{2}x=1$

Sol

$$\text{L.H.S}\ =\left(\frac{(e^{x}-e^{-x})}{e^{x}+e^{-x}}\right)^{2}+\left(\frac{2}{(e^{x}+e^{-x})}\right)^{2}$$

$$=\frac{e^{2x}-2e^{x}e^{-x}+e^{-2x}}{e^{2x}+2e^{x}e^{-x}+e^{-2x}}+\frac{4}{e^{2x}+2e^{x}e^{-x}+e^{-2x}}$$

$$=\frac{e^{2x}-2+e^{-2x}+4}{e^{2x}+2+e^{-2x}}=\frac{e^{2x}+2+e^{-2x}}{e^{2x}+2+e^{-2x}}=1\ \text{R.H.S}$$

EXAMPLE 5: Prove that $\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$

Sol:

L.H.S $\sinh(x+y) = \dfrac{e^{(x+y)} - e^{-(x+y)}}{2}$

R.H.S $\sinh x \cosh y + \cosh x \sinh y$

$$= \left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{y} + e^{-y}}{2}\right) + \left(\frac{e^{x} + e^{-x}}{2}\right)\left(\frac{e^{y} - e^{-y}}{2}\right)$$

$$= \frac{1}{4}(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)}) + \frac{1}{4}(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)})$$

$$= \frac{1}{4}e^{x+y} + \frac{1}{4}e^{x+y} - \frac{1}{4}e^{-(x+y)} - \frac{1}{4}e^{-(x+y)}$$

$$= \frac{1}{2}e^{x+y} - \frac{1}{2}e^{-(x+y)}$$

$$= \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \text{L.H.S}$$

EXAMPLE 6: Rewrite the following expressions in terms of exponential. Write the final result as simply as you can.

1. $2\cosh(\ln x)$

   Sol:

$$2\cosh(\ln x) = 2\left(\frac{e^{\ln x} + e^{-\ln x}}{2}\right)$$

$$= e^{\ln x} + e^{-\ln x}$$

$$= x + 1/x$$

$$2\cosh(\ln x) = x + 1/x$$

2. $\tanh(\ln x)$

   Sol:

$$\tanh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}}$$

$$= \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{\frac{x^2 - 1}{x}}{\frac{x^2 + 1}{x}}$$

$$\tanh(\ln x) = \frac{x^2 - 1}{x^2 + 1}$$

EXAMPLE 7: use the definitions the identity $\cosh^2 x - \sinh^2 x = 1$to find the values of the remaining five hyperbolic function. $\sinh = -3/4$

Sol:

1. $\cosh^2 x = 1 + \sinh^2 x$

   $\cosh x = \sqrt{1 + \sinh^2 x}$

   $\cosh x = \sqrt{1 + (-3/4)^2} = \sqrt{25/16} = 5/4$

2. $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{-3/4}{5/4} = -3/5$

3. $\coth x = \dfrac{1}{\tanh x} = \dfrac{-5}{3}$

4. $\text{sechx} = \dfrac{1}{\cosh x} = \dfrac{4}{5}$

5. $\text{cschx} = \dfrac{1}{\sinh x} = \dfrac{-4}{3}$

H.W **Ex 8:**

1. $\sinh = 4/3$
2. $\cosh = 17/15$
3. $\cosh = 13/5$

EXAMPLE 9: Rewrite the expressions (1 - 5) in terms of exponential and simply the result as much as you can.

1. $\sinh(2\ln x)$

   Sol:

$$\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2}$$

$$= \frac{x^2 - x^{-2}}{2} = \frac{x^4 - 1}{2x^2}$$

$2.\ \cosh 5x + \sinh 5x$ H.W

$3.\ \cosh 3x - \sinh 3x$ H.W

$4.\ (\sinh x + \cosh x)^4$

Sol:

$$(\sinh x + \cosh x)^4 = (\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2})^4$$

$$= (e^x)^4 = e^{4x}$$

$5.\ \ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

Sol:

$$\ln(\cosh x + \sinh x)(\cosh x - \sinh x)$$

$$\ln(\cosh^2 x - \sinh^2 x)$$

$$\ln 1 = 0$$

EXAMPLE 10: Prove that $\sinh^{-1} x = \ln(x + \sqrt{1 + x^2})$

Sol:

Let $y = \sinh^{-1} x$ L.H.S

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y} \qquad \times\ e^y$$

$$2xe^y = e^{2y} - 1$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$Ax^2 + Bx + c = 0$$

$$B = -2x$$

$$A = 1$$

$$C = -1$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$e^y = x \pm \frac{2\sqrt{x^2+1}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$\ln e^y = \ln(x \pm \sqrt{x^2+1})$$

$$y = \ln(x \pm \sqrt{x^2+1})$$

$$y = \ln(x + \sqrt{x^2+1})$$

H.W **Ex 11:** Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$

EXAMPLE 12: Prove that $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Sol:

Let $y = \tanh^{-1} x$

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$xe^y + xe^{-y} = e^y - e^{-y} \quad \times \quad e^y$$

$$xe^{2y} + x = e^{2y} - 1$$

$$xe^{2y} - e^{2y} = -x - 1$$

$$(x-1)e^{2y} = -(x+1)$$

$$e^{2y} = \frac{-(x+1)}{(x-1)}$$

$$\ln e^{2y} = \ln \frac{-(x+1)}{(x-1)}$$

$$2y = \ln \frac{-(x+1)}{-(1-x)} \quad \div 2$$

$$y = \frac{1}{2} \ln \frac{(x+1)}{(1-x)} \qquad \text{R.H.S}$$

EXAMPLE 13: Prove that $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

Sol:

Let $y = \operatorname{sech}^{-1} x$

$$x = \operatorname{sech} y$$

$$x = \frac{2}{e^y + e^{-y}}$$

$$xe^y + xe^{-y} = 2 \qquad \times e^y$$

$$xe^{2y} + x = 2e^y$$

$$xe^{2y} - 2e^y + x = 0$$

$$e^y = \frac{-(-2) \pm \sqrt{(-2)^2 - (2x)^2}}{2x}$$

$$e^y = \frac{2}{2x} \pm \frac{\sqrt{4-4x^2}}{2x}$$

$$e^y = \frac{1}{x} \pm \frac{2\sqrt{1-x^2}}{2x}$$

$$e^y = \frac{1}{x} \pm \frac{\sqrt{1-x^2}}{x}$$

$$e^y = \left(\frac{1 \pm \sqrt{1-x^2}}{x}\right)$$

$$\ln e^{y} = \ln(\frac{1 \pm \sqrt{1-x^2}}{x})$$

$$y = \ln(\frac{1 + \sqrt{1-x^2}}{x})$$

EXAMPLE 14: Use the identities

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

To show that

a) $\sinh 2x = 2\sinh x \cosh x$

b) $\cosh 2x = \cosh^2 x + \sinh^2 x$

Sol: a

$$\sinh(2x) = \sinh(x+x)$$
$$= \sinh x \cosh x + \cosh x \sinh x$$
$$= 2\sinh x \cosh x$$

Sol: b

$$\cosh(2x) = \cosh(x+x)$$
$$= \cosh x \cosh x + \sinh x \sinh x$$
$$= \cosh^2 x + \sinh^2 x$$