Chapter three

Derivatives

<u>Rule of Derivatives:</u> Let c and n are constant, u, v and ware differentiable function of x:

1. $\frac{d}{dx}c = 0$
2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
3. $\frac{d}{dx}(\frac{1}{u}) = -\frac{1}{u^2}\frac{du}{dx}$
4. $\frac{d}{dx}cu = c\frac{du}{dx}$
5. $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$ and $\frac{d}{dx}(u \cdot v \cdot w) = u \cdot v\frac{dw}{dx} + u \cdot w\frac{dv}{dx} + v \cdot w\frac{du}{dx}$
6. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where $v \neq 0$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = (x^2 + 1)^5$

Sol: $y = (x^2 + 1)^5$

$y' = 5(x^2 + 1)^4(2x)$

$y' = 10x(x^2 + 1)^4$

2. $y = \frac{x^2 - 1}{x^2 + x - 2}$

Sol:

$$y' = \frac{2x(x^2 + x - 2) - (2x + 1)(x^2 - 1)}{(x^2 + x - 2)^2}$$

$$y' = \frac{2x^3 + 2x^2 - 4x - 2x^3 + 2x - x^2 + 1}{(x^2 + x - 2)^2} = \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2}$$

H.W **Ex 3:** $y = \frac{12}{x} - \frac{4}{x^2} + \frac{3}{x^4}$

H.W **Ex 4**: $y = (2x^3 - 3x^2 + 6x)^{-5}$

H.W **Ex 5**: $y = \frac{x^2 - 1}{x^2 + x - 2}$

EXAMPLE 6: $y = \frac{x^2 - 1}{x + 1}$

Sol:

$$y' = \frac{(x+1)(2x) - (x^2 - 1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2 + 1}{(x+1)^2} = \frac{x^2 + 2x + 1}{(x+1)^2}$$

EXAMPLE 7: $y = \sqrt[3]{x^2} \quad \Rightarrow \quad y = x^{2/3}$

$$y' = \frac{2}{3} x^{-1/3}$$

The chain rule

1. Suppose that $h = g \cdot f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable function of x whose derivative at each value of x is

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}}$$

EXAMPLE 1: Find $\frac{dy}{dx}$ if $y = \frac{1}{t^2 + 1}$, $x = \sqrt{4t + 1}$

Sol:

$$y = (t^2 + 1)^{-1}, \; x = \sqrt{4t + 1}$$

$$\frac{dy}{dx}=\frac{dy}{dt}/\frac{dx}{dt}=\frac{\frac{d}{dt}(t^2+1)^{-1}}{\frac{d}{dt}(4t+1)^{1/2}}$$

$$=\frac{-(t^2+1)^{-2}(2t)}{\frac{1}{2}(4t+1)^{-1/2}\cdot 4}=\frac{-2t\,(t^2+1)^{-2}}{2(4t+1)^{-1/2}}$$

$$=\frac{-t\,(t^2+1)^{-2}}{(4t+1)^{-1/2}}$$

2. If y is a differentiable function of t and t is a differentiable function of x ,then y is a differentiable of x:

$y=g(t)$ and $t=f(x)$

$$\boxed{\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}}$$

EXAMPLE 1: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y

$$y=\frac{t^2}{t^2+1} \quad , \quad t=\sqrt{2x+1}=(2x+1)^{1/2}$$

Sol:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{(t^2+1)2t-t^2(2t)}{(t^2+1)^2}\cdot\frac{1}{2}(2x+1)^{-1/2}(2)$$

$$=\frac{2t^3+2t-2t^3}{(t^2+1)^2}\cdot(2x+1)^{-1/2}$$

$$=\frac{2t}{(t^2+1)^2}\cdot\frac{1}{\sqrt{2x+1}} \qquad \text{sub } t$$

$$=\frac{2\sqrt{2x+1}}{(2x+1+1)^2}\cdot\frac{1}{\sqrt{2x+1}}=\frac{2}{(2x+2)^2}$$

EXAMPLE 2: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y

$$y = \left(\frac{t-1}{t+1}\right)^2, \; x = \frac{1}{t^2} - 1 \quad \text{at} \quad t = 2$$

Sol:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \Rightarrow \qquad y = \left(\frac{t-1}{t+1}\right)^2$$

$$\frac{dy}{dt} = 2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{4(t-1)}{(t+1)^3}$$

$$\left.\frac{dy}{dt}\right|_{t=2} = \frac{4(2-1)}{(2+1)^3} = 4/27$$

$$x = \frac{1}{t^2} - 1$$

$$\left.\frac{dx}{dt}\right|_{t=2} = \frac{-2}{t^3} = -1/4$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 4/27 \div (-1/4) = -16/27$$

Higher derivative

If a function $y = f(x)$ possesses a derivative at every point of some interval. We may form the function $f'(x)$ and take about its derivate if it has one.

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x)$$

This derivative is called the second derivative of y with respect to x. In some manner we may define third and higher derivatives using similar notations.

EXAMPLE 1: Find all derivatives of the following function.

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol:

$$y' = 9x^2 - 8x + 7$$

$$y'' = 18x - 8$$

$$y''' = 18$$

$$y'''' = 0$$

EXAMPLE 2:

$$y = \frac{1}{x} + \sqrt{x^3} \quad \Rightarrow y = x^{-1} + x^{3/2}$$

Sol:

$$y' = -\frac{1}{x^2} + \frac{3}{2} x^{1/2}$$

$$y'' = \frac{2}{x^3} + \frac{3}{4} x^{-1/2}$$

$$y''' = -\frac{6}{x^4} - \frac{3}{8} x^{-3/2}$$

$$= -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

Implicit derivative

If the formula of f is an algebraic combination of power of x and y.To calculate the derivative of the implicitly defined functions. We simply differentiable both sides of the defining equation with respect to x.

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $x^2y^2 = x^2 + y^2$

Sol:

$$x^2 2yy' + 2xy^2 = 2x + 2y\,y'$$
$$x^2 2yy' - 2y\,y' = 2x - 2xy^2$$
$$y'(2x^2y - 2y) = 2x - 2xy^2$$
$$y' = \frac{2x - 2xy^2}{2x^2y - 2y} = \frac{x - xy^2}{x^2y - y}$$

2. $\frac{x-y}{x-2y} = 2$

Sol:

$$2x - 4y = x - y$$
$$2 - 4y' = 1 - y'$$
$$2 - 1 = -y' + 4y'$$
$$1 = 3y'$$
$$y' = 1/3$$

3. $xy + 2x - 5y = 2$ at $(3,2)$

Sol:

$$xy' + y + 2 - 5y' = 0$$
$$y'(x-5) = -y - 2$$

$$y' = \frac{-(y+2)}{(x-5)} = \frac{-(2+2)}{(3-5)} = \frac{-4}{-2} = 2$$

EXAMPLE 2: write an equation for the tangent line at $x=3$ of the curve $f(x) = \frac{1}{\sqrt{2x+3}}$

Sol:

Equation of tangent line $y - y_1 = m\,(x - x_1)$

$y = \frac{1}{\sqrt{2x+3}}$ at $x=3$ $\Rightarrow$ $y = \frac{1}{\sqrt{9}} = \frac{1}{3}$

P $(3, 1/3)$

The slope of y' $\Rightarrow$ $\frac{dy}{dx} = \frac{d}{dx}(2x+3)^{-1/2}$

$y' = -1/2\,(2x+3)^{-3/2}\;(2)$

$y' = \frac{-1}{(2x+3)^{3/2}}$ y' at $x=3$ $= \frac{-1}{(2(3)+3)^{3/2}} = -\frac{1}{9^{3/2}}$

$y' = -\frac{1}{3^{2^{3/2}}} = -\frac{1}{3^3} = -\frac{1}{27}$

$y' = -\frac{1}{27}$

$(y - 1/3) = (-1/27)\;(x-3)$ Equation of tangent

$y - 1/3 = -1/27\,x + 1/9$

$y = -1/27\,x + 1/9 + 1/3$

$y = -1/27\,x + 4/9$

$y + 1/27\,x - 4/9 = 0$

Trigonometric function

1. $\sin u$ $\qquad \frac{d}{x}\sin u = \cos u \frac{du}{dx}$
2. $\cos u$ $\qquad \frac{d}{x}\cos u = -\sin u \frac{du}{dx}$
3. $\tan u$ $\qquad \frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$
4. $\cot u$ $\qquad \frac{d}{x}\cot u = -\csc^2 u \frac{du}{dx}$
5. $\sec u$ $\qquad \frac{d}{x}\sec u = \sec u \tan u \frac{du}{dx}$
6. $\csc u$ $\qquad \frac{d}{x}\csc u = -\csc u \cot u \frac{du}{dx}$

EXAMPLE 1: Prove that $\frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\tan u = \frac{d}{x}\frac{\sin u}{\cos u}$$
$$= \frac{\cos u \cos u - \sin u(-\sin u)}{\cos^2 u}\frac{du}{dx}$$
$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx}$$
$$= \frac{1}{\cos^2 u}\frac{du}{dx} = \sec^2 u \frac{du}{dx}$$

EXAMPLE 2: Prove that $\frac{d}{x}\sec u = \sec u \tan u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\sec u = \frac{d}{dx}\frac{1}{\cos u} = -\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$

$$= \frac{\sin u}{\cos u}\frac{1}{\cos u}\frac{du}{dx}$$

$$= \sec u \tan u \frac{du}{dx}$$

EXAMPLE 3: Find $\frac{dy}{dx}$ for the following function

1. $y = \tan(3x^2)$

Sol:

$$\frac{dy}{dx} = \sec^2(3x^2)(6x) = 6x\sec^2(3x^2)$$

2. $y = (\csc x + \cot x)^2$

Sol:

$$\frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x \cot x - \csc^2 x)$$

$$= -2\csc x(\csc x + \cot x)^2$$

Hint:

1. $y = \sin^n u$ $\qquad$ $y' = n\sin^{n-1} u \cos u \frac{du}{dx}$
2. $y = \cos^n u$ $\qquad$ $y' = n\cos^{n-1} u(-\sin u)\frac{du}{dx}$
3. $y = \tan^n u$ $\qquad$ $y' = n\tan^{n-1} u \sec^2 u \frac{du}{dx}$
4. $y = \cot^n u$ $\qquad$ $y' = n\cot^{n-1} u(-\csc^2 u)\frac{du}{dx}$
5. $y = \sec^n u$ $\qquad$ $y' = n\sec^{n-1} u(\sec u \tan u)\frac{du}{dx}$
6. $y = \csc^n u$ $\qquad$ $y' = n\csc^{n-1} u(-\csc u \cot u)\frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = \tan^2(\cos x)$

$$y' = 2\tan(\cos x)\sec^2(\cos x)(-\sin x)$$

$$y' = -2\sin x\tan(\cos x)\sec^2(\cos x)$$

$$y = \sec^4 x - \tan^4 x$$

$$y' = 4\sec^3 x \sec x \tan x - 4\tan^3 x \sec^2 x$$

$$= 4\sec^4 x \tan x - 4\tan^3 x \sec^2 x$$

$$= \sec^2 x(4\sec^2 x \tan x - 4\tan^3 x)$$

2. $= 2\tan(x/2) - x$

$$y' = 2\sec^2(x/2)\cdot(1/2) - 1$$

$$= \sec^2(x/2) - 1$$

$$y' = \tan^2(x/2)$$

3. $y = \cot^3 x$

$$y' = 3\cot^2 x(-\csc^2 x)\cdot 1$$

$$y' = -3\cot^2 x \csc^2 x$$

4. $x + \tan(xy) = 0$

$$1 + \sec^2(xy)(xy' + y)$$

$$\sec^2(xy)xy' + \sec^2(xy)y = -1$$

$$xy' \sec^2(xy) = -(1 + y\sec^2(xy))$$

$$y' = \frac{-(1 + y\sec^2(xy))}{x\sec^2(xy)}$$

5. $y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$

$$y' = 2\cos\frac{x}{2}\cdot\frac{1}{2} - (x(-\sin\frac{x}{2})\cdot\frac{1}{2} + \cos\frac{x}{2})$$

$$= \cos\frac{x}{2} + \frac{x}{2}\sin\frac{x}{2} - \cos\frac{x}{2} = \frac{x}{2}\sin\frac{x}{2}$$

Transcendental function derivative

1- Logarithm function الدالة الوغارتيمية

(1) If $y = \ln x$ $\quad \boxed{\Rightarrow \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}}$

EXAMPLE: $y = \ln x$

$$y' = \frac{1}{x}$$

(2) $\boxed{\frac{d}{dx}\log_a u = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}}$

EXAMPLE: $y = \log_a x = \frac{\ln x}{\ln a}$ $\quad \Rightarrow \quad y' = \frac{1}{\ln a}\frac{1}{x}$

(3) If $y = \log x = \frac{\ln x}{\ln 10}$ $\quad \Rightarrow y' = \frac{1}{\ln 10}\frac{1}{x}$

$$\boxed{\frac{d}{dx}\log u = \frac{1}{u}\frac{1}{\ln 10}\frac{du}{dx}}$$

EXAMPLE 1: $y = \ln(\sin x - \sec x)$

$$y' = \frac{\cos x - \sec x \tan x}{\sin x - \sec x}$$

EXAMPLE 2: Find $\frac{dy}{dx}$ for the following function:

1. $y = \log_{10} e^x$

Sol:

$$y = x\log_{10} e \quad \Rightarrow \quad y' = \log_{10} e$$

2. $y = \log_5 (x+1)^2$

Sol:

$$y = 2\log_5 (x+1) \quad \Rightarrow \quad y = 2\frac{\ln(x+1)}{\ln 5}$$

$$y' = 2\frac{1}{(x+1)\ln 5}$$

3. $y = \log_2 (3x^2+1)^3$

Sol:

$$y = 3\log_2 (3x^2+1) \quad \Rightarrow \quad y = 3\frac{\ln(3x^2+1)}{\ln 2}$$

$$y' = 3\frac{6x}{(3x^2+1)\ \ln 2} = \frac{18x}{(3x^2+1)\ \ln 2}$$

4. $y + \ln x + \ln y = 1$ find y'

Sol:

$$y' + \frac{1}{x} + \frac{1}{y}y' = 0$$

$$y'(1+\frac{1}{y}) = -\frac{1}{x}$$

$$y'(\frac{y+1}{y}) = -\frac{1}{x}$$

$$y' = -\frac{y}{x(y+1)}$$

5. $\sin(\ln y) = \ln(x^2 - 3x + 1)$

Sol:

$$\cos(\ln y)\cdot\frac{1}{y}y' = \frac{2x-3}{x^2-3x+1}$$

$$\cos(\ln y)\,y' = \frac{y(2x-3)}{x^2-3x+1}$$

$$y' = \frac{y(2x-3)}{\cos(\ln y)(x^2-3x+1)}$$

EXAMPLE 6: If $y=\ln(t)$ $\quad t=\ln(x^2-1)$

Sol:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dt}\ln(t)\cdot\frac{d}{dx}\ln(x^2-1)$$

$$y' = \frac{1}{t}\cdot\frac{2x}{(x^2-1)}$$

$$t=\ln(x^2-1)$$

$$y' = \frac{1}{\ln(x^2-1)}\cdot\frac{2x}{(x^2-1)}$$

ملاحظة يمكن استخدام الدالة الوغارتيمية في ايجاد مشتقة الدوال المعقدة او الدوال الاسية

EXAMPLE 7: Find $\frac{dy}{dx}$ for the following function:

1. $y = x^x$

Sol:

$$\ln y = \ln x^x$$

$$\ln y = x\ln x$$

$$\frac{1}{y}y' = x\frac{1}{x}+\ln x \quad\Rightarrow y' = y(1+\ln x)$$

2. $y = x^{\tan x}$

Sol:

$$\ln y = \tan x \ln x$$

$$\frac{1}{y} y' = \tan x \, \frac{1}{x} + \sec^2 x \ln x$$

$$y' = y \ \left(\frac{\tan x}{x} + \sec^2 x \ln x\right)$$

3. $y = \sin x \tan x \cos x \sec x \cot x$

Sol:

$$\ln y = \ln(\sin x \tan x \cos x \sec x \cot x)$$

$$\ln y = \ln \sin x + \ln \tan x + \ln \cos x + \ln \sec x + \ln \cot x$$

$$\frac{1}{y} y' = \frac{\cos x}{\sin x} + \frac{\sec^2 x}{\tan x} + \frac{-\sin x}{\cos x} + \frac{\sec x \tan x}{\sec x} + \frac{-\csc^2 x}{\cot x}$$

$$y' = y(\cot x + \frac{\sec^2 x}{\tan x} - \tan x + \tan x - \frac{\csc^2 x}{\cot x})$$

$$y' = y(\cot x + \sec^2 x \cot x - \csc^2 x \tan x)$$

4. $y = \sqrt[3]{\frac{x \sin x}{(x-1)(x^2+1)}} \qquad \Rightarrow \qquad y = \left(\frac{x \sin x}{(x-1)(x^2+1)}\right)^{1/3}$

Sol:

$$\ln y = \frac{1}{3} \ln \frac{x \sin x}{(x-1)(x^2+1)}$$

$$\ln y = \frac{1}{3}[\ln x + \ln \sin x - (\ln(x-1) + \ln(x^2+1))]$$

$$\frac{1}{y} y' = \frac{1}{3}[\frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1}{(x-1)} - \frac{2x}{(x^2+1)}]$$

$$y' = \frac{y}{3}(\frac{1}{x} + \cot x - \frac{1}{(x-1)} - \frac{2x}{(x^2+1)})$$

2- **Exponential function** If is u any differentiable function of x then:

1) $\frac{d}{dx} = a^u = a^u \ln a \frac{du}{dx}$

2) $\frac{d}{dx} = e^u = e^u \frac{du}{dx}$

EXAMPLE 7: Find $\frac{dy}{dx}$ for the following function:

1. $y = 2^{3x}$

   $y' = 2^{3x} 3 \ln 2$

2. $y = (2^x)^2 \quad \Rightarrow \quad y = 2^{2x}$

   $y' = 2^{2x} \ln 2(2) = 2^{2x+1} \ln 2$

3. $y = x 2^{x^2}$

   $y' = x 2^{x^2} \ln 2(2x) + 2^{x^2} = 2^{x^2}(2x^2 \ln 2 + 1)$

4. $y = e^{\sqrt{1-5x^2}} = e^{(1-5x^2)^{1/2}}$

   $y' = e^{(1-5x^2)^{1/2}} \cdot \frac{1}{2}(1-5x^2)^{-1/2}(10x) = e^{(1-5x^2)^{1/2}} \frac{5x}{\sqrt{1-5x^2}}$

5. $y = e^{7x}$

   $y' = 7e^{7x}$

6. $y = e^{\tan x}$

$y' = e^{\tan x} \sec^2 x$

7. $y = 3^{\tan x}$

$y' = \ln 3 \, 3^{\tan x} \sec^2 x$

8. $y = x 2^{x^2}$

$y' = x \ln 2 \; 2^{x^2} (2x) + 2^{x^2}$

9. $e^{(x+y)} = \ln(x^2 + y^2) + \sin x + \tan x$

$$e^{(x+y)}(1 + y') = \frac{2x + 2y\,y'}{x^2 + y^2} + \cos x + \sec^2 x$$

$$e^{(x+y)} + e^{(x+y)} y' = \frac{2x}{x^2 + y^2} + \frac{2y\,y'}{x^2 + y^2} + \cos x + \sec^2 x$$

$$e^{(x+y)} y' - \frac{2y\,y'}{x^2 + y^2} = \frac{2x}{x^2 + y^2} + \cos x + \sec^2 x - e^{(x+y)}$$

$$y'(e^{(x+y)} - \frac{2y}{x^2 + y^2}) = \frac{2x}{x^2 + y^2} + \cos x + \sec^2 x - e^{(x+y)}$$

$$y' = \frac{\frac{2x}{x^2 + y^2} + \cos x + \sec^2 x - e^{(x+y)}}{(e^{(x+y)} - \frac{2y}{x^2 + y^2})}$$

10. $y^x = x^y$

$\ln y^x = \ln x^y$

$x \ln y = y \ln x$

$$x \frac{y'}{y} + \ln y = y \frac{1}{x} + \ln x \, y'$$

$$\frac{x}{y}y' - y'\ln x = \frac{y}{x} - \ln y$$

$$y'(\frac{x}{y} - \ln x) = \frac{y}{x} - \ln y$$

$$y' = \frac{\frac{y}{x} - \ln y}{\frac{x}{y} - \ln x}$$

Inverse function

1. **Trigonometric function**

(1) $\frac{d}{x}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(2) $\frac{d}{x}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(3) $\frac{d}{x}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

(4) $\frac{d}{x}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$

(5) $\frac{d}{x}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u > 1$

(6) $\frac{d}{x}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u > 1$

EXAMPLE 1: Prove that $\frac{d}{x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

Proof:

Let $y = \sin^{-1}x$

$x = \sin y$

$$\frac{d}{dx}x = \frac{d}{x}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \sqrt{1-\sin^2 y}$$

$$\cos y = \sqrt{1-x^2}$$

EXAMPLE 2: Prove that $\frac{d}{x}\tan^{-1} x = \frac{1}{1+x^2}$

Proof:

Let $y = \tan^{-1} x$

$$x = \tan y$$

$$\frac{d}{dx} x = \frac{d}{x}(\tan y)$$

$$1 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$1 + \tan^2 y = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

EXAMPLE 3: If $y = \tan^{-1}(x^2 - x)$ find $\frac{dy}{dx}$

Sol:

$$y' = \frac{1}{1+u^2} = \frac{2x-1}{1+(x^2-x)^2}$$

EXAMPLE 4: If $y = \sin^{-1} \ln(x)$ find y'

Sol:

$$y' = \frac{1}{\sqrt{1-u^2}} = \frac{1/x}{\sqrt{1-(\ln x)^2}}$$

EXAMPLE 5: $y = e^{\tan^{-1}(3x)}$ find y'

Sol:

$$y' = e^{\tan^{-1}(3x)} \frac{3}{1+(3x)^2}$$

EXAMPLE 6: $y = \ln(e^{\sin^{-1}x} - \tan^{-1}x)$

Sol:

$$y' = \frac{e^{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}} - \frac{1}{1+x^2}}{e^{\sin^{-1}x} - \tan^{-1}x}$$

EXAMPLE 7: If $y = \cot^{-1} 2/x + \tan^{-1} x/2$

Sol:

$$y' = \frac{-(-2/x^2)}{1+(2/x)^2} + \frac{1/2}{1+(x/2)^2}$$

$$y' = \frac{2/x^2}{1+4/x^2} + \frac{1/2}{1+(x/2)^2}$$

EXAMPLE 8: If $y = \sin^{-1}(\frac{x-1}{x+1})$ find y'

Sol:

$$y' = \frac{\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}}{\sqrt{1-(\frac{x-1}{x+1})^2}} = \frac{\frac{x+1-x+1}{(x+1)^2}}{\sqrt{1-(\frac{x-1}{x+1})^2}}$$

$$y' = \frac{2/(x+1)^2}{\sqrt{1-(\frac{x-1}{x+1})^2}}$$

EXAMPLE 9: If $y = \sec^{-1}(5x)$ find y'

Sol:

$$y' = \frac{5}{5x\sqrt{25x^2 - 1}}$$

EXAMPLE 10: If $y = x \cdot \ln \sec^{-1} x$ find $\frac{dy}{dx}$

Sol:

$$y' = x \frac{\frac{1}{x\sqrt{x^2 - 1}}}{\sec^{-1} x} + \ln(\sec^{-1} x)$$

$$y' = \frac{1}{\sqrt{x^2 - 1}\ \sec^{-1} x} + \ln(\sec^{-1} x)$$

EXAMPLE 11: $y = 3^{\sin^{-1}(2x)} \Rightarrow$ find y'

Sol:

$$y' = \ln 3\ \ 3^{\sin^{-1}(2x)} \frac{2}{\sqrt{1 - 4x^2}}$$

Hyperbolic function

If u is any differentiable function of x

1. $\frac{d}{x} \sinh u = \cosh u \frac{du}{dx}$
2. $\frac{d}{x} \cosh u = \sinh u \frac{du}{dx}$
3. $\frac{d}{x} \tanh u = \text{sech}^2 u \frac{du}{dx}$

4. $\frac{d}{x}\coth u = -\operatorname{csch}^2 \frac{du}{dx}$

5. $\frac{d}{x}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

6. $\frac{d}{x}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function:

1. $y = \coth(\tan x)$

Sol:

$$y' = -\operatorname{csch}^2(\tan x)\sec^2 x$$

2. $y = \sin^{-1}(\tanh x)$

Sol:

$$y' = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \operatorname{sech} x$$

3. $y = \ln \tanh x/2$

Sol:

$$y' = \frac{\operatorname{sech}^2 x/2 \cdot (1/2)}{\tanh x/2} = \frac{1}{2}\frac{\frac{1}{\cosh^2 x/2}}{\frac{\sinh x/2}{\cosh x/2}}$$

$$y' = \frac{1}{2\sinh x/2 \cdot \cosh x/2} = \frac{1}{\sinh x} = \operatorname{csch} x$$

4. $y = x\sinh 2x - \frac{1}{2}\cosh 2x$

Sol:

$$y' = x\cosh 2x \cdot 2 + \sinh 2x - \tfrac{1}{2}\sinh 2x \cdot 2$$

$$y' = 2x\cosh 2x$$

5. $y = \text{sech}^3 x$

Sol:

$$y' = 3\text{sech}^2 x(-\text{sech}x \cdot \tanh x)$$

$$y' = -3\text{sech}^3 x \ \tanh x)$$

6. $y = \text{csch}^2 x$

Sol:

$$y' = 2\text{csch}x(-\text{csch}x \coth x)$$

$$y' = -2\text{csch}^2 x \coth x$$

EXAMPLE 2: Prove that $\frac{d}{x}\tanh u = \text{sech}^2 u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\tanh u = \frac{d}{x}\left(\frac{\sinh u}{\cosh u}\right)$$

$$= \frac{\cosh u \ \cosh u \frac{du}{dx} - \sinh u \sinh u \frac{du}{dx}}{\cosh^2 u}$$

$$= \frac{(\cosh^2 u - \sinh^2 u)\ \frac{du}{dx}}{\cosh^2 u} = \frac{1}{\cosh^2 u}\frac{du}{dx}$$

$$\frac{d}{x}\tanh u = \text{sech}^2 u \frac{du}{dx}$$

EXAMPLE 3: Prove that $\frac{d}{x}\text{sech}u = -\text{sech}u \tanh u \frac{du}{dx}$

Sol:

$$= \frac{d}{dx}\frac{1}{\cosh u} = -\frac{1}{\cosh^2 u}\sinh u \frac{du}{dx}$$

$$= -\operatorname{sech} u \tanh u \frac{du}{dx}$$

EXAMPLE 4: Show that the functions

$$x = -\frac{2}{\sqrt{3}} \sinh(t/\sqrt{3}) \qquad y = \frac{1}{\sqrt{3}} \sinh(t/\sqrt{3}) + \cosh(t/\sqrt{3})$$

Taken together, satisfy the differential equation

I. $\frac{dx}{dt} + 2\frac{dy}{dt} + x = 0$

II. $\frac{dx}{dt} - \frac{dy}{dt} + y = 0$

Proof: I

$$\frac{dx}{dt} = -\frac{2}{\sqrt{3}} \cosh(t/\sqrt{3}) \frac{1}{\sqrt{3}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{3}} \cosh(t/\sqrt{3}) \frac{1}{\sqrt{3}} + \sinh(t/\sqrt{3}) \frac{1}{\sqrt{3}}$$

I. $\frac{-2}{3} \cosh(t/\sqrt{3}) + 2\,\frac{1}{3} \cosh(t/\sqrt{3}) + \frac{2}{\sqrt{3}} \sinh(t/\sqrt{3}) + \frac{-2}{\sqrt{3}} \sinh(t/\sqrt{3}) = 0$

II. $\frac{-2}{3} \cosh(t/\sqrt{3}) - \frac{1}{3} \cosh(t/\sqrt{3}) - \frac{1}{\sqrt{3}} \sinh(t/\sqrt{3}) + \frac{1}{\sqrt{3}} \sinh(t/\sqrt{3}) + \cosh(t/\sqrt{3}) = 0$

<u>The Inverse hyperbolic function</u> If is u any differentiable function of x then:

1. $\frac{d}{x}\sinh^{-1}u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

2. $\frac{d}{x}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

3. $\frac{d}{x}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx} \qquad u<1$

4. $\frac{d}{x}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx} \qquad u>1$

5. $\frac{d}{x}\mathrm{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$

6. $\frac{d}{x}\mathrm{csch}^{-1}u = -\frac{-1}{u\,\sqrt{1+u^2}}\frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = \cosh^{-1}(\sec x)$

Sol:

$$y' = \frac{\sec x \tan x}{\sqrt{(\sec x)^2 - 1}} = \frac{\sec x \tan x}{\sqrt{\tan^2 x}}$$

$$y' == \frac{\sec x \tan x}{\tan x} = \sec x \qquad \text{where } \tan x > 0$$

2. $y = \tan\text{h}^{-1}(\cos x)$

Sol:

$$y' = \frac{-\sin x}{1 - \cos^2 x} = \frac{-\sin x}{\sin^2 x} = -\csc x$$

3. $y = \cot\text{h}^{-1}(\sec x)$

Sol:

$$y' = \frac{\sec x \tan x}{1 - \sec^2 x} = \frac{\sec x \tan x}{-\tan^2 x} = -\csc x$$

4. $y = \text{sech}^{-1}(\sin 2x)$

Sol:

$$y' = -\frac{2\cos 2x}{\sin 2x\sqrt{1 - \sin^2 2x}} = \frac{-2\cos 2x}{\sin 2x \cos 2x} = -2\csc 2x \qquad \text{where } 2x > 0$$

EXAMPLE 2: Verify the following formulas:

1. $\dfrac{d}{x}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2 - 1}}\dfrac{du}{dx}$

Sol:

Let $y = \cosh^{-1} u$

$$u = \cosh y$$

$$\frac{du}{dx} = \sinh y \frac{dy}{dx}$$

$$y' = \frac{dy}{dx} = \frac{1}{\sinh y}\frac{du}{dx}$$

$$\cosh^2 y - \sinh^2 y = 1 \qquad \Rightarrow u^2 - \sinh^2 y = 1$$

$$\Rightarrow \sinh y = \sqrt{u^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

2. $\frac{d}{x} \tanh^{-1} u = \frac{1}{1-u^2} \frac{du}{dx}$

Sol:

Let $y = \tanh^{-1} u$

$u = \tanh y$

$$\frac{du}{dx} = \text{sech}^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\text{sech}^2 y} \frac{du}{dx}$$

$$\text{sech}^2 y + \tanh^2 y = 1 \quad \Rightarrow \text{sech}^2 y = 1 - \tanh^2 y = 1 - u^2$$

$$\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$$

$$\Rightarrow \frac{d}{x} \tanh^{-1} u = \frac{1}{1-u^2} \frac{du}{dx}$$

مراجعة

((اسئلة اضافية))

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = mx + b$

Sol:

$$y' = m$$

2. $y = \frac{1}{x}$

Sol:

$$y' = \frac{-1}{x^2}$$

3. $y = \frac{x}{x-1}$

Sol:

$$y' = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

4. $y = \sqrt{x}$

Sol:

$$y = x^{1/2} \Rightarrow y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

EXAMPLE 2: Find the value of the derivative.

1. $\left.\frac{ds}{dt}\right|_{t=1}$ if $s=1-3t^2$

Sol:

$$\frac{ds}{dt}=0-6t \qquad \left.\frac{ds}{dt}\right|_{t=1}=-6(-1)=6$$

2. $\left.\frac{dy}{dx}\right|_{x=\sqrt{3}}$ $\quad y=1-\frac{1}{x} \quad \Rightarrow \; y=1-x^{-1}$

Sol:

$$\frac{dy}{dx}=0-(-1)(x^{-2})$$

$$\frac{dy}{dx}=\frac{1}{x^2} \qquad \Rightarrow \; \left.\frac{dy}{dx}\right|_{x=\sqrt{3}}=\frac{1}{(\sqrt{3})^2}=\frac{1}{3}$$

3. $\left.\frac{dr}{d\theta}\right|_{\theta=0}$ $\quad$ if $r=\frac{2}{\sqrt{4-\theta}} \quad \Rightarrow \; r=2(4-\theta)^{-1/2}$

Sol:

$$\frac{dr}{d\theta}=-\frac{1}{2}(2)(4-\theta)^{-3/2}(-1)$$

$$\left.\frac{dr}{d\theta}\right|_{\theta=0}=\frac{1}{4^{3/2}}=8$$

EXAMPLE 3: Find the derivative of $y=\frac{t^2-1}{t^2+1}$

Sol:

$$\frac{dy}{dt}=\frac{(t^2+1)(2t)-(t^2-1)(2t)}{(t^2+1)^2}$$

$$\frac{dy}{dt}=\frac{4t}{(t^2+1)^2}$$

EXAMPLE 4: Find an equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1,3)$

Sol:

$$\frac{dy}{dt} = 1 + \frac{-2}{x^2}$$

The slope at $x = 1$

$$y'\big|_{x=1} = [1 - \frac{2}{x^2}]_{x=1}$$

$$= 1 - 2 = -1$$

$$m = -1$$

The line through $(1,3)$ with slope $m = -1$

$$y - 3 = (-1)(x - 1)$$

$$y = -x + 1 + 3$$

$$y = -x + 4$$

EXAMPLE 5: Find higher derivatives $y = x^3 - 3x^2 + 2$

Sol:

First $y' = 3x^2 - 6x$

Second $y'' = 6x - 6$

Third $y''' = 6$

Fourth $y'''' = 0$

EXAMPLE 6: Find $\frac{dy}{dx}$ for the following Trigonometric function.

1. $y = \frac{\sin x}{x}$

Sol:

$$y' = \frac{x\cos x - \sin x(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

2. $y = x^2 \sin x$

Sol:

$$y' = x^2 \cos x + 2x \sin x$$

3. $y = \sin x \cos x$

Sol:

$$y' = \sin x(-\sin x) + \cos x \cos x$$

$$y' = \cos^2 x - \sin^2 x$$

4. $y = \dfrac{\cos x}{1 - \sin x}$

Sol:

$$y' = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2}$$

$$y' = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2}$$

$$= \frac{1}{1 - \sin x}$$

5. If $y = \sec^2 x$ find y''

Sol:

$$y' = \sec x \tan x$$

$$y'' = \sec x \sec^2 x + \sec x \tan x \tan x$$

$$y'' = \sec^3 x + \sec x \tan^2 x$$

6. Find the slope of the line tangent to the curve $y = \sin^5 x$ at point where $x = \pi / 3$

Sol:

$$\frac{dy}{dt} = 5 \sin^4 x \cos x$$

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4 \cdot \frac{1}{2} = \frac{45}{32}$$

$\cos\pi/3 = 1/2$

$\sin\pi/3 = \sqrt{3}/2$

7. If $x = 2t+3$ and $y = t^2 - 1$ Find the value of $\frac{dy}{dx}$ at $t = 6$

Sol:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t$$

$$\frac{dy}{dx} = 6 \qquad t = 6$$

Hint: Note that we are also able to find $\frac{dy}{dx}$ as a function of x

$x = 2t+3 \qquad y = t^2 - 1$

$x - 3 = 2t$

$t = \frac{x-3}{2}$

8. Find $\frac{dy}{dx}$ if $y^2 = x$

Sol:

$$2y\,y' = 1$$

$$y' = \frac{1}{2y}$$

9. Find the slope of circle $x^2 + y^2 = 25$ at the point $(3,-4)$

Sol:

$$2x + 2y\,y' = 0$$

$$y' = \frac{-2x}{2y} \quad \Rightarrow \quad y' = -\frac{x}{y}$$

$$\left.y'\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$$

10. Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Sol:

$$2y\,y' = 2x + \cos xy\,(x\,y' + y)$$

$$2y\,y' = 2x + x\,y'\cos xy + y\cos xy$$

$$2y\,y' - x\,y'\cos xy = 2x + y\cos xy$$

$$y'(2y - x\cos xy) = 2x + y\cos xy$$

$$y' = \frac{2x + y\cos xy}{2y - x\cos xy}$$

11. Find $\frac{d}{dx}(\cos x)^{-1/5}$

Sol:

$$-\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$

$$\frac{1}{5}\sin x(\cos x)^{-6/5}$$

12. Find $\frac{d}{dx}\ln 2x$

Sol:

$$y' = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}\cdot 2 = \frac{1}{x}$$

13.Find $\frac{d}{dx}\ln(x^2+3)$

Sol:

$$y' = \frac{1}{x^2+1}\cdot(2x) = \frac{2x}{x^2+1}$$

14.Find $\frac{d}{dx}\ln x^r$

Sol:

$$y' = \frac{1}{x^r}\cdot rx^{r-1}$$

$$y' = \frac{1}{x^r} r\, x^r\, x^{-1} = \frac{r}{x}$$

15.Find $\frac{dy}{dx}$ if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1} \qquad x>1$

Sol: we take natural logarithm of the both side

$$\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$\ln y = \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1)$$

$$\ln y = \ln(x^2+1) + \tfrac{1}{2}\ln(x+3) - \ln(x-1)$$

$$\frac{y'}{y} = \frac{2x}{(x^2+1)} + \frac{1}{2}\frac{1}{(x+3)} - \frac{1}{(x-1)}$$

$$y' = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

16. Find $\frac{dy}{dx}$ if 1) $y = 5e^x$ 2) $y = e^{-x}$ 3) $y = e^{\sin x}$

Sol:

1) $y' = 5e^x$
2) $y' = -e^{-x}$
3) $y' = e^{\sin x}\cos x$

17.Find $\frac{dy}{dx}$ if 1) $y = x^{\sqrt{2}}$ 2) $y = (2+\sin 3x)^{\pi}$

Sol:

1) $y' = \sqrt{2}\,x^{\sqrt{2}-1}$
2) $y' = \pi(2+\sin 3x)^{\pi-1}(\cos 3x)\cdot 3$

   $y' = 3\pi(2+\sin 3x)^{\pi-1}(\cos 3x)$

18. Find $\frac{dy}{dx}$ if 1) $y = 3^x$ 2) $y = 3^{-x}$ 3) $y = 3^{\sin x}$

Sol:

1) $y' = 3^x \ln 3$
2) $y' = -3^x \ln 3$
3) $y' = 3^{\sin x}(\ln 3)\cos x$

19. Find $\frac{dy}{dx}$ if $\quad y = x^x \quad x > 0$

Sol:

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{y'}{y} = (x\frac{1}{x} + \ln x)$$

$$y' = y\ (1 + \ln x)$$

$$y' = x^x\ (1 + \ln x)$$

Or write x^x as a power of e

$$y = x^x = e^{x \ln x}$$

$$\frac{dy}{dx} = \frac{d}{dx} e^{x \ln x}$$

$$\frac{dy}{dx} = e^{x \ln x} (\frac{d}{dx} x \ln x)$$

$$\frac{dy}{dx} = x^x\ (x\frac{1}{x} + \ln x)$$

$$= x^x\ (1 + \ln x)$$

20. Find $\frac{d}{dx} \log_{10}(3x+1)$

Sol:

$$\frac{d}{dx} \log_{10}(3x+1)$$

$$\frac{d}{dx} \frac{\ln(3x+1)}{\ln 10}$$

$$\frac{1}{\ln 10} \frac{3}{3x+1}$$

21. If $\sinh u = \frac{e^u - e^{-u}}{2}$ find $\frac{d}{dx} \sinh u$

Sol:

$$\frac{d}{dx}\sinh u = \frac{e^u(1) - e^{-u}(-1)}{2}$$

$$= \frac{e^u + e^{-u}}{2}$$

$$= \cosh u$$

22. Find $\frac{d}{dt}\tanh\sqrt{1+t^2}$

Sol:

$$= \operatorname{sech}^2\sqrt{1+t^2}\ \frac{d}{dt}(1+t^2)^{1/2}$$

$$= \operatorname{sech}^2\sqrt{1+t^2}\ \frac{1}{2}\frac{1}{\sqrt{1+t^2}}\cdot 2t$$

$$= \operatorname{sech}^2\sqrt{1+t^2}\ \ \frac{t}{\sqrt{1+t^2}}$$

Hint:

1) $\operatorname{sech}^{-1}x = \cosh^{-1}(1/x)$
2) $\operatorname{csch}^{-1}x = \sinh^{-1}(1/x)$
3) $\coth^{-1}x = \tanh^{-1}(1/x)$

23. Find y' or $\frac{dy}{dx}$ $\sin^{-1}(xy) = \cos^{-1}(x-y)$

Sol:

$$\frac{xy'+y}{\sqrt{1-(xy)^2}} = \frac{-(1-y')}{\sqrt{1-(x-y)^2}}$$

$$\frac{xy'}{\sqrt{1-(xy)^2}} + \frac{y}{\sqrt{1-(xy)^2}} = \frac{-1}{\sqrt{1-(x-y)^2}} + \frac{y'}{\sqrt{1-(x-y)^2}}$$

$$y'[\frac{x}{\sqrt{1-(xy)^2}} - \frac{1}{\sqrt{1-(x-y)^2}}] = -\frac{1}{\sqrt{1-(x-y)^2}} - \frac{y}{\sqrt{1-(xy)^2}}$$

$$y' = \frac{-\frac{1}{\sqrt{1-(x-y)^2}} - \frac{y}{\sqrt{1-(xy)^2}}}{\frac{x}{\sqrt{1-(xy)^2}} - \frac{1}{\sqrt{1-(x-y)^2}}}$$

$$y' = \frac{\frac{-\sqrt{1-(xy)^2} - y\sqrt{1-(x-y)^2}}{\sqrt{1-(x-y)^2}\sqrt{1-(xy)^2}}}{\frac{x\sqrt{1-(x-y)^2} - \sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2}\sqrt{1-(x-y)^2}}}$$

$$y' = \frac{-\sqrt{1-(xy)^2} - y\sqrt{1-(x-y)^2}}{x\sqrt{1-(x-y)^2} - \sqrt{1-(xy)^2}}$$

24. Show for $y = \frac{U}{V}$ that $y'' = \frac{V(VU'' - UV'') - 2V'(VU' - UV')}{V^3}$

Sol:

$$y = \frac{U}{V}$$

$$y'' = \frac{VU' - UV'}{V^2}$$

$$y'' = \frac{V^2(VU'' + V'U' - UV'' - U'V') - (VU' - UV')2VV'}{V^4}$$

$$= \frac{V^2(VU'' + V'U' - UV'' - U'V') - 2V'(V^2U' - UVV')}{V^4}$$

$$= \frac{V[V(VU'' - UV'') - 2V'(VU' - UV')]}{V^4}$$

$$= \frac{V(VU'' - UV'') - 2V'(VU' - UVV')}{V^3}$$

25.Show that $y = 35x^4 - 30x^2 + 3$ satisfies $(1-x^2)y'' - 2xy' + 20y = 0$

Sol:

$$y' = 140x^3 - 60x \qquad y'' = 420x^2 - 60$$

$$(1-x^2)(420x^2-60)-2x(140x^3-60x)+20(35x^4-30x^2+3)$$

$$420x^2-60-420x^4+60x^2-280x^4+120x^2+700x^4-600x^2+60$$

$$420x^2+60x^2+120x^2-600x^2-420x^4-280x^4+700x^4-60+60=0$$

H.W Derivative

1) Find $\frac{dy}{dx}$ for the following function

1. $y=\csc^{-2/3}\sqrt{5x}$ — **ans:** $\frac{5}{3\sqrt{5x}}\cdot\frac{\cot\sqrt{5x}}{\csc^{2/3}\sqrt{5x}}$

2. $y=(x-3)(1-x)$ — **ans:** $4-2x$

3. $y=\frac{ax+b}{x}$ — **ans:** $-\frac{b}{x^2}$

4. $y=\ln(\cos x)$ — **ans:** $-\tan x$

5. $y=\tan x\sin x$ — **ans:** $\sin x+\tan x\sec x$

6. $y=\frac{3x-4}{2x-3}$ — **ans:** $\frac{1}{(2x+3)^2}$

7. $y=(\sqrt{x^3}-\frac{1}{\sqrt{x}})^2$ — **ans:** $\frac{3(x^5-1)}{x^4}$

8. $y=\frac{\cos x}{x}$ — **ans:** $\frac{-x\cdot\sin x+\cos x}{x^2}$

9. $y=\tan(\sec x)$ — **ans:** $\sec^2(\sec x)\sec x\cdot\tan x$

10. $y=x^2\sin x$ — **ans:** $x^2\cos x+2x\cdot\sin x$

11. $y = \sin^{-1}(5x^2)$ **ans:** $\frac{10x}{\sqrt{1-25x^4}}$

12. $y = \cot^3(\frac{x+1}{x-1})$ **ans:** $\frac{6}{(x-1)^2}\cot^2(\frac{x+1}{x-1})\csc^2(\frac{x+1}{x-1})$

13. $y = \sin(\ln\ x) + \cos(\ln x)$ **ans:** $2\cos(\ln x)$

14. $y = \cot^{-1}(\frac{1+x}{1-x})$ **ans:** $-\frac{1}{1+x^2}$

15. $y = \tan^{-1}\sqrt{4x^3-2}$ **ans:** $-\frac{6x^2}{(4x^3-1)\sqrt{4x^3-2}}$

16. $y = \sec^{-1}(3x^2+1)^3$ **ans:** $\frac{18x}{(3x^2+1)\sqrt{(3x^2+1)^6-1}}$

17. $y = \sin^{-1}2x\cos^{-1}2x$ **ans:** $\frac{2(\cos^{-1}2x - \sin^{-1}2x)}{\sqrt{1-4x^2}}$

18. $y = \tan^{-1}\ln x$ **ans:** $\frac{1}{x(1+(\ln x)^2)}$

19. $y = (\cos x)^{\sqrt{x}}$ **ans:** $\frac{y}{2\sqrt{x}}(\ln\cos x - 2x\tan x)$

20. $y = (\sin x)^{\tan x}$ **ans:** $y(1+\sec^2 x\ln\sin x)$

21. $y = \sqrt{2x^2+\cosh^2(5x)}$ **ans:** $\frac{2x+5\cosh(5x)\cdot\sinh(5x)}{\sqrt{2x^2+\cosh^2(5x)}}$

22. $y = \sinh(\cos 2x)$ **ans:** $-2\sin 2x\cosh(\cos 2x)$

23. $y = \csc(1/x)$ **ans:** $\frac{1}{x^2}\cdot\csc(1/x)\cdot\cot(1/x)$

24. $y = x^2\cdot\tanh^2\sqrt{x}$ **ans:** $x\cdot\tanh\sqrt{x}(\sqrt{x}\,\mathrm{sech}^2\sqrt{x} + 2\tanh\sqrt{x})$

25. $y = \ln\frac{\sin x\cos x + \tan^3 x}{\sqrt{x}}$ **ans:** $\frac{\cos^2 x - \sin^2 x + 3\tan^2 x}{\sin x\cos x + \tan^3 x} - \frac{1}{2x}$

26. $y = \log_4 \sin x$ **ans:** $\dfrac{\cot x}{\ln 4}$

27. $y = e^{(x^2 - e^{5x})}$ **ans:** $(2x - 5e^{5x})e^{(x^2 - e^{5x})}$

28. $y = e^{x^2 \tan x}$ **ans:** $(x^2 \sec^2 x + 2x \tan x)e^{x^2 \tan x}$

29. $y = 7^{\csc\sqrt{2x+3}}$ **ans:** $\dfrac{-7^{\csc\sqrt{2x+3}} \ln 7 \csc\sqrt{2x+3} \cot\sqrt{2x+3}}{\sqrt{2x+3}}$

30. $y = [\ln(x^2 + 2)^2]\cos x$ **ans:** $\dfrac{4x\cos x}{x^2 + 2} - 2\ln(x^2 + 2)\sin x$

31. $y = \sinh^{-1}(\tan x)$ **ans:** $\sec x$

32. $y = \sqrt{1 + (\ln x)^2}$ **ans:** $\dfrac{\ln x}{x\sqrt{1 + (\ln x)^2}}$

33. $\dfrac{e^x}{\ln x}$ **ans:** $\dfrac{e^x (\ln x - 1)}{x (\ln x)^2}$

34. $x^3 \log_2(3 - 2x)$ **ans:** $2x^2 \log_2(3 - 2x) - \dfrac{2x^3}{(3 - 2x)\ln 2}$

35. $y = 2\cosh^{-1}\dfrac{x}{2} + \dfrac{x}{2}\sqrt{x^2 - 4}$ **ans:** $\dfrac{x^2}{\sqrt{x^2 - 4}}$

2) Verify the following derivative

1. $\dfrac{d}{dx}[5x + (\sqrt{x} + \dfrac{1}{\sqrt{x}})^2 = 6 - \dfrac{1}{x^2}$

2. $\dfrac{d}{dx}[\sqrt{x} + (ax^2 + bx + c)] = \dfrac{1}{2\sqrt{x}}(5ax^2 + 3bx + c)$

3) Find the derivative of y with respect to x in the following function:

1. $y = \dfrac{u^2}{u^2 + 1}$ and $u = 3x^2 - 2$ **ans:** $\dfrac{18x^2 y^2}{(3x^3 - 2)^3}$

2. $y=\sqrt{u}+2u$ and $u=x^2-3$ **ans:** $\dfrac{x}{\sqrt{x^2-3}}+4x$

4) Find the second derivative for the following function

1. $y=(x+\dfrac{1}{x})^3$ **ans:** $6x+\dfrac{6}{x^3}+\dfrac{12}{x^5}$

2. $y=\sqrt{2x}+\dfrac{2\sqrt{2}}{\sqrt{x}}$ at $x=2$ **ans:** $\dfrac{1}{4}$

3. $y=x^2-2xy+y^2-16x=0$ **ans:** $\pm x^{-3/2}$

5) Find the third derivative of the function $y=\sqrt{x^3}$ **ans:** $-\dfrac{3}{8y}$

6) Show that $y=\dfrac{U}{V}$ that $y''=\dfrac{V(VU''-UV'')-2V'(VU'-UV')}{V^3}$

7) Show that $y=35x^4-30x^2+3$ satisfies $(1-x^2)y''-2xy'+20y=0$

8) Find $\dfrac{dy}{dx}$ for the following implicit function:

1. $\sqrt{xy}+1=y$ **ans:** $\dfrac{y}{2\sqrt{xy}-x}$

2. $\sinh y=\tan^2 x$ **ans:** $\dfrac{2\tan x\cdot\sec^2 x}{\cosh y}$

3. $x^3+4x\sqrt{y}-\dfrac{5y^2}{x}=3$ **ans:** $\dfrac{3x^2+5y^2x^{-2}+4\sqrt{y}}{10x^{-1}y-(2x/\sqrt{y})}$

4. $3xy=(x^3+y^3)^{3/2}$ **ans:** $\dfrac{3x^2\sqrt{x^3+y^3}-2y}{2x-3y^2\sqrt{x^3+y^3}}$

5. $\sin^{-1}(xy)=\cos^{-1}(x-y)$ **ans:** $\dfrac{y\sqrt{1-(x-y)^2}+\sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2}-x\sqrt{1-(x-y)^2}}$

6. $y^2 \sin(xy) = \tan x$ **ans:** $\dfrac{\sec^2 x - y^3 \cos(xy)}{2y\sin(xy) + xy^2 \cos(xy)}$

7. $x^3 + x\tan^{-1} y = y$ **ans:** $\dfrac{(1+y^2)(3x^2 + \tan^{-1} y)}{1+y^2 - x}$

9) Prove the following function

1. $\dfrac{d}{x}\cot u = -\csc^2 u \dfrac{du}{dx}$

2. $\dfrac{d}{x}\csc u = -\csc u \cot u \dfrac{du}{dx}$

3. $\dfrac{d}{x}\cosh^{-1} u = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

4. $\dfrac{d}{x}\operatorname{sec}h^{-1} u = \dfrac{1}{|u|\sqrt{u^2 - 1}}\dfrac{du}{dx}$

5. $\dfrac{d}{x}\sinh u = \cosh u \dfrac{du}{dx}$

6. $\dfrac{d}{x}\operatorname{csch} u = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

7. $\dfrac{d}{x}\sinh^{-1} u = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$

8. $\frac{d}{x}\operatorname{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$