Chapter five

Integration

Integration is the reversal of differentiation hence functions can be integrated by indentifying the anti-derivative.

Terminology

Indefinite and **Definite** integrals

There are two types of integrals: Indefinite and Definite.

Indefinite integrals are those with no limits and definite integrals have limits.

When dealing with indefinite integrals you need to add a constant of integration.

For example, if integrating the function f(x) with respect to x:

$$\int f(x)\ dx = g(x) + C$$

where $g(x)$ is the integrated function.

C is an arbitrary constant called the constant of integration.

dx indicates the variable with respect to which we are integrating, in this case, *x*.

The function being integrated, f(*x*), is called the **integrand**.

The Rule

1) **Constant Rule** $\boxed{\int a\,dx = ax + c}$ where a is constant

EXAMPLE:

1. $\int 3\,dx = 3x + c$
2. $\int 4\,dy = 4y + c$
3. $\int \frac{7}{2}\,dz = \frac{7}{2}z + c$

2) **Sum Rule** $\boxed{\int (f \pm g)\,dx = \int f\ dx \pm \int g\,dx}$

EXAMPLE:

1. $\int 3\,dx + 4dy = \int 3\,dx + \int 4dy = 3x + 4y + c$

3) The Power Rule $n \neq -1$

$$\int ax^n du = a\frac{x^{n+1}}{n+1} + c$$

EXAMPLE: **1.** $\int 4x^5 dx = 4\frac{x^6}{6} + c = \frac{2}{3}x^6 + c$

2. $\int 10x^{-5} dx = 10\frac{x^{-4}}{-4} + c = -\frac{5}{2}x^{-4} + c$

4) The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

EXAMPLES: **1.** $\int (x+1)^3 dx = \frac{(x+1)^4}{4} + c$

2. $\int (x^2+1)^4\, 2x\,dx = \frac{(x^2+1)^5}{5} + c$

3. $\int (x^3-4)^7\, 3x^2\, dx = \frac{(x^3-4)^8}{8} + c$

4. $\int (x^2-2x-4)^7\,(2x-2)\,dx = \frac{(x^2-2x-4)^8}{8} + c$

5. $\int (x^2+2)^5\, x\,dx = \frac{1}{2}\int (x^2+2)^5\, 2x\,dx$

$$= \frac{1}{2}\frac{(x^2+2)^6}{6} + c$$

6. $\int (x^2+2x+3)^7\,(x+1)\,dx = \frac{1}{2}\int (x^2+2x+3)^7\, 2(x+1)\,dx$

$$= \frac{1}{2}\int (x^2+2x+3)^7\,(2x+2)\,dx = \frac{1}{2}\frac{(x^2+2x+3)^8}{8}$$

Root function integral

EXAMPLES: **1.** $\int 2x\sqrt{x^2-3}\,dx = \int (x^2-3)^{1/2}\,2x\,dx$

$$\frac{(x^2-3)^{3/2}}{3/2}+c$$

$$\frac{2}{3}(x^2-3)^{3/2}+c$$

2. $\int x\sqrt{3x^2+1}\,dx = \frac{1}{6}\int (3x^2+1)^{1/2}\,6x\;dx$

$$=\frac{2}{3}\frac{(3x^2+1)^{3/2}}{6}+c=\frac{1}{9}(3x^2+1)^{3/2}+c$$

3. $\int \frac{x-1}{\sqrt{x^2-2x-3}}\,dx = \frac{1}{2}\int (x^2-2x-3)^{-1/2}\,(2x-2)\,dx$

$$=\frac{1}{2}\frac{(x^2-2x-3)^{1/2}}{1/2}+c$$

$$=\sqrt{(x^2-2x-3)}+c$$

4. $\int (x^3-1)^2\,x\,dx = \int (x^6-2x^3+1)\,x\,dx$

$$=\int (x^7-2x^4+x)\;dx$$

$$=\frac{x^8}{8}-\frac{2}{5}x^5+\frac{1}{2}x+c$$

5. $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx$

$$=\int \frac{(\sqrt{x})^2-2\sqrt{x}+1}{\sqrt{x}}\,dx$$

$$=\int \left(\frac{x}{\sqrt{x}}-2\frac{\sqrt{x}}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx$$

$$=\int (\sqrt{x}-2+\frac{1}{\sqrt{x}})\,dx$$

$$=\int (x^{1/2}\,dx-\int 2\,dx+\int x^{-1/2}dx$$

$$=\frac{x^{3/2}}{3/2}-2x+\frac{x^{1/2}}{1/2}+c=\frac{2}{3}x^{3/2}-2x+2x^{1/2}+c$$

Or $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx=2\int (\sqrt{x}-1)^2\,\frac{1}{2\sqrt{x}}dx=2\frac{(\sqrt{x}-1)^3}{3}+c$

6. $\int \frac{1}{x}\sqrt{x^2-x^3}\,dx$

$$=\int \frac{1}{x}\sqrt{x^2(1-x)}\,dx$$

$$=\int \frac{x}{x}\sqrt{(1-x)}\,dx$$

$$=\int \sqrt{(1-x)}\,\,dx$$

$$=-\int -(1-x)^{1/2}\,dx$$

$$=-\frac{(1-x)^{3/2}}{3/2}+c=-\frac{2}{3}(1-x)^{3/2}+c$$

7. $\int \frac{dx}{(4x^2-12x+9)^{3/2}}\,\,dx$

$$=\int (4x^2-12x+9)^{-3/2}\,dx$$

$$=\int ((2x-3)^2)^{-3/2}\,dx$$

$$=\int (2x-3)^{-3}\,dx$$

$$=\frac{1}{2}\int (2x-3)^{-3}\,2\,dx$$

$$=\frac{1}{2}\,\frac{(2x-3)^{-2}}{-2}+c=-\frac{1}{4}(2x-3)^{-2}+c$$

H.W **Evaluate**

1. $= \int \sqrt{x^2 - x^4}\, dx$

2. $\int (x^2+1)^2 (x+2)\, dx$

3. $\int \frac{2x-4}{\sqrt{x^2-4x+1}}\, dx$

تكامل الدوال المثلثية

<u>Trigonometric function integral</u> $\sin u$, $\cos u$

1. $\sin u$ $\qquad \frac{d}{x}\sin u = \cos u \frac{du}{dx}$

$$\boxed{\int \cos u\, du = \sin u + c}$$

2. $\cos u$ $\qquad \frac{d}{x}\cos u = -\sin u \frac{du}{dx}$

$$\boxed{\int \sin u\, du = -\cos u + c}$$

EXAMPLES:

1. $\int \cos 3x\, dx = \frac{\sin 3x}{3} + c$

2. $\int \sin 7x\, dx = \frac{-\cos 7x}{7} + c$

3. $\int \sin(3x-1)\, dx = \frac{-\cos(3x-1)}{3} + c$

اذا كانت الدالة اسية ومشتقة داخل القوس متوفرة عندها نستخدم القوانين التالية

1. $$\int \sin^n au \cos au\, du = \frac{\sin^{n+1} au}{(n+1)a} + c$$

2. $$\int \cos^n au \sin au\, du = \frac{-\cos^{n+1} au}{(n+1)a} + c$$

EXAMPLE:

1. $$\int \sin^7 3x \cos 3x\, dx = \frac{\sin^8 3x}{(8)(3)} + c$$

2. $$\int \sin 2x \cos^3 2x\, dx = -\frac{\cos^4 2x}{(4)\,(2)} + c$$

3. $$\int \sin x \cos x\, dx = \int (\sin^1 x) \cos x\, dx = \frac{\sin^2 x}{2a} + c$$

اذا كانت الدالة اسية والمشتقة غير متوفرة نتبع مايلي

1. اذا كانت الدالة اسية والمشتقة غير متوفرة وكان الاس عدد زوجي

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

EXAMPLES: **1.**

$$\int \sin^2 x\, dx = \int (\frac{1}{2} - \frac{1}{2}\cos 2x)\, dx$$

$$= \int \frac{1}{2} dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$$

2. $$\int \cos^2 3x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2(3x)\right) dx$$

$$= \int \frac{1}{2} dx + \frac{1}{2}\int \cos 6x\, dx$$

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin 6x}{6} + c$$

2. اذا كانت الاس عدد فردي نستخدم القانون

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

EXAMPLE 1 $$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$$

$$= \int (1 - \cos^2 x) \sin x\, dx$$

$$= \int (\sin x - \cos^2 x \sin x)\, dx$$

$$= \int \sin x\, dx - \int \cos^2 x \sin x\, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

EXAMPLE 2 $$\int \cos^3 2x\, dx = \int \cos^2 2x \cos 2x\, dx$$

$$= \int (1 - \sin^2 2x) \cos 2x\, dx$$

$$= \int (\cos 2x - \sin^2 2x \cos 2x)\, dx$$

$$= \int \cos 2x\, dx - \int \sin^2 2x \cos 2x\, dx$$

$$= \frac{\sin 2x}{2} - \frac{\sin^3 2x}{3(2)} + c$$

EXAMPLE 3: $\int \sin^4 x \quad dx = \int (\sin^2 x)^2\, dx$

$$= \int (\frac{1}{2} - \frac{1}{2}\cos 2x)^2\, dx$$

$$= \int (\frac{1}{4} - \frac{1}{4} 2\cos 2x + \frac{1}{4}\cos^2 2x)\, dx$$

$$= \int \frac{1}{4} dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{4}\int \cos^2 2x \; dx$$

$$= \frac{1}{4}x - \frac{1}{2}\frac{\sin 2x}{2} + \frac{1}{4}\int (\frac{1}{2} + \frac{1}{2}\cos 4x)\, dx$$

$$= \frac{1}{4}x - \frac{1}{2}\frac{\sin 2x}{2} + \frac{1}{4}\frac{1}{2}x + \frac{1}{4}\frac{1}{2}\frac{\sin 4x}{4} + c$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{\sin 4x}{32} + c$$

EXAMPLE 4: $\int \cos^4 3x \quad dx = \int (\cos^2 3x)^2\, dx$

$$= \int (\frac{1}{2} + \frac{1}{2}\cos 6x)^2\, dx$$

$$= \int (\frac{1}{4} + \frac{1}{2}\cos 6x + \frac{1}{4}\cos^2 6x)\, dx$$

$$= \int \frac{1}{4} dx + \frac{1}{2}\int \cos 6x\, dx + \frac{1}{4}\int \cos^2 6x \; dx$$

$$= \frac{1}{4}x + \frac{1}{2}\frac{\sin 6x}{6} + \frac{1}{4}\int (\frac{1}{2} + \frac{1}{2}\cos 12x)\, dx$$

$$=\frac{1}{4}x+\frac{1}{12}\sin 6x+\frac{1}{4}(\frac{1}{2}x+\frac{1}{2}\frac{\sin 12x}{12})+c$$

EXAMPLE 5: $\int \sin^5 x\,dx=\int(\sin^2 x)^2\sin x\,dx$

$$=\int(1-\cos^2 x)^2\ \sin x\,dx$$

$$=\int(1-2\cos^2 x+\cos^4 x)\ \sin x\ dx$$

$$=\int(\sin x-2\cos^2 x\ \sin x+\cos^4 x\ \sin x)\ dx$$

$$=-\cos x+\frac{2\cos^3 x}{3}-\frac{\cos^5 x}{5}+c$$

اذا كان السؤال حاصل ضرب دالتين عندها نتبع الفوانين التالية

1. اولا الاسس فردية عندها نفك الاس الفردي الاقل مرتبة ونحل حسب القوانين الفردية

EXAMPLE : $\int \sin^3 x\cos^5 x\ dx=\int \sin^2 x\ \sin x\cos^5 x\,dx$

$$=\int(1-\cos^2 x)\ \sin x\cos^5 x\,dx$$

$$=\int(\sin x\cos^5 x-\cos^7 x\ \sin x)\,dx$$

$$=\int \cos^5\ \sin x-\int \cos^7 x\ \sin x\,dx$$

$$=-\frac{\cos^6 x}{6}+\frac{\cos^8 x}{8}+c$$

2. اذا كان احد الاس فردي والاخر زوجي عندها نفك الاس الفردي حسب القوانين الفردية

EXAMPLE : $\int \sin^5 x\cos^2 x\ dx=\int(\sin^2 x)^2\ \sin x\cos^2 x\,dx$

$$=\int (1-\cos^2 x)^2 \ \sin x \cos^2 x\, dx$$

$$=\int (1-2\cos^2 x+\cos^4 x)\ \sin x\ \cos^2 x\, dx$$

$$=\int \sin x \cos^2 x\ dx-\int 2\cos^4 x \sin x\, dx+\int \cos^6 x \sin x\ dx$$

$$=-\frac{\cos^3 x}{3}+2\frac{\cos^5 x}{5}-\frac{\cos^7 x}{7}+c$$

اذا كان السؤال حاصل ضرب دالتين والزوايا مختلفة عندها نستخدم القوانين التالية

$\int \sin mx \sin nx\ dx$ $\int \sin mx \cos nx\ dx$ $\int \cos mx \cos nx\ dx$

1. $\sin mx \sin nx=\frac{1}{2}\cos(m-n)\,x-\frac{1}{2}\cos(m+n)\,x$
2. $\sin mx \cos nx=\frac{1}{2}\sin(m-n)\,x+\frac{1}{2}\sin(m+n)\,x$
3. $\cos mx \cos nx=\frac{1}{2}\cos(m-n)\,x+\frac{1}{2}\cos(m+n)\,x$

Evaluate

1. $\int \sin 7x \cos x\ dx$

$$=\int \frac{1}{2}(\sin 6x+\sin 8x)\ dx$$

$$=-\frac{1}{2}\frac{\cos 6x}{6}-\frac{1}{2}\frac{\cos 8x}{8}+c$$

$$=-\frac{1}{12}\cos 6x-\frac{1}{16}\cos 8x+c$$

2. $\int \sin 7x \cos 3x\, dx$

$$=\int \tfrac{1}{2}\sin(7-3)\,x\, dx+\int \tfrac{1}{2}\sin(7+3)\,x\, dx$$

$$=\int \tfrac{1}{2}\sin 4\,x\, dx+\int \tfrac{1}{2}\sin 10\,x\, dx$$

$$= -\frac{1}{2}\frac{\cos 4x}{4} + \frac{1}{2}\frac{(-\cos 10x)}{10} + c$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + c$$

3. $\int \cos 2x \cos 3x \, dx$

$$= \int \tfrac{1}{2}(\cos(3-2)x + \cos(3+2)x)\, dx$$

$$= \frac{1}{2}\int \cos x \, dx + \frac{1}{2}\int \cos 5x \, dx$$

$$= \frac{1}{2}\sin x + \frac{1}{2}\frac{\sin 5x}{5} + c$$

ثانيا تكاملات

Integral $\tan x$, $\cot x$, $\sec x$ and $\csc x$

- اولا تكاملات مباشرة

1. If $y = \tan u \;\rightarrow\; y' = \sec^2 u \dfrac{du}{dx}$

$$\boxed{\int \sec^2 u \, du = \tan u + c}$$

2. If $y = \cot u \;\rightarrow\; y' = -\csc^2 u \dfrac{du}{dx}$

$$\boxed{\int \csc^2 u \, du = -\cot u + c}$$

3. If $y = \sec u \;\rightarrow\; y' = \sec u \tan u \dfrac{du}{dx}$

$$\boxed{\int \sec u \tan u \; du = \sec u + c}$$

4. If $y = \csc u \;\rightarrow\; y' = -\csc u \cot u \dfrac{du}{dx}$

$$\boxed{\int \csc u \cot u \; du = -\csc u + c}$$

- ثانيا اذا كانت الدالة اسية المشتقة متوفرة عندها نستخدم القونين التالية

$$\int \tan^n au \sec^2 au\, du = \frac{\tan^{n+1} au}{(n+1)a} + c$$

$$\int \cot^n au \csc^2 au\, du = -\frac{\cot^{n+1} au}{(n+1)a} + c$$

$$\int \sec^n au \sec au \tan au\, du = \frac{\sec^{n+1} au}{(n+1)a} + c$$

$$\int \csc^n au \csc au \cot au\, du = -\frac{\csc^{n+1} au}{(n+1)a} + c$$

EXAMPLES:

Find the following integral

1) $\int \tan x \sec^2 x\, dx = \frac{\tan^2 x}{2} + c$

2) $\int \tan^2 x \sec^2 x\, dx = \frac{\tan^3 x}{3} + c$

3) $\int \tan^7 x \sec^2 x\, dx = \frac{\tan^8 x}{8} + c$

4) $\int \frac{\sqrt{\tan x - 1}}{\cos^2 x} dx = \int \frac{(\tan x - 1)^{1/2}}{\cos^2 x} dx$

$= \int (\tan x - 1)^{1/2} \sec^2 x\, dx$

$$=\frac{(\tan x-1)^{3/2}x}{3/2}+c$$

- ثالثا اذا كانت الدالة اسية المشتقة غيرمتوفرة عندها نستخدم القونين التالية

$$\sec^2 x-\tan^2 x=1$$
$$\tan^2 x=\sec^2 x-1$$
$$\sec^2 x=1+\tan^2 x$$

$$\csc^2 x-\cot^2 x=1$$
$$\csc^2 x=1+\cot^2 x$$
$$\cot^2 x=\csc^2 x-1$$

EXAMPLE 1: $\int \tan^2 x\,dx=\int (\sec^2 x-1)dx$

$$=\int \sec^2 x\,dx-\int dx$$

$$=\tan x-x+c$$

EXAMPLE 2: $\int \sec^4 x\,dx=\int \sec^2 x\sec^2 x\,dx$

$$=\int (1+\tan^2 x)\sec^2 x\,dx$$

$$=\int \sec^2 x\,dx+\int \tan^2 x\ \sec^2 x\,dx$$

$$=\tan x+\frac{\tan^3 x}{3}+c$$

Transcendental Function Integral

Logarithm **exponential** **invers**

(1) If $y = \ln u \quad \Rightarrow y' = \dfrac{du}{u}$

$y = \ln x \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$

$$\boxed{\int \frac{du}{u} = \ln|u| + c}$$

بمعنى ان مشتقة البسط = المقام فان التكامل هو **ln** المقام

EXAMPLE 1: $\int \dfrac{dx}{x} = \ln|x| + c$

2. $\int \dfrac{2x\,dx}{x^2+1} = \ln\left|x^2+1\right| + c$

3. $\int \dfrac{(x+1)\,dx}{x^2+2x+3} = \dfrac{1}{2}\int \dfrac{(2x+2)\,dx}{x^2+2x+3} = \dfrac{1}{2}\ln\left|x^2+2x+3\right| + c$

4. $\int \dfrac{x^2\,dx}{1-x^3} = \dfrac{3}{3}\int \dfrac{x^2\,dx}{1-x^3}$

$= \dfrac{-1}{3}\int \dfrac{-3x^2\,dx}{1-x^3} = \dfrac{-1}{3}\ln\left|1-x^3\right| + c$

5. $\int \dfrac{\sin x\,dx}{1-\cos x} = -\ln\left|1-\cos x\right| + c$

6. $\int \frac{\sec^2 x\,dx}{3+\tan x} = \ln|3+\tan x| + c$

7. $\int \frac{(x-2)\,dx}{x^2-4x-3} = \frac{1}{2}\int \frac{(2x-4)\,dx}{x^2-4x-3} = \frac{1}{2}\ln\left|x^2-4x-3\right| + c$

8. $\int \frac{\cos x\,dx}{1+\sin x} = \ln|1+\sin x| + c$

9. $\int \frac{\sin 2x\,dx}{\sin^2 x} = \ln\left|\sin^2 x\right| + c$

Sol: let $u = \sin^2 x$, $du = 2\sin x\cos x\,dx$

$$\sin 2x = 2\sin x\cos x$$

$$\sin x\cos x = \tfrac{1}{2}\sin(x-x) + \tfrac{1}{2}\sin(x+x)$$

$$= \tfrac{1}{2}\sin(2x)$$

$$\therefore\ du = \tfrac{1}{2}2\sin(2x) = \sin 2x$$

$$\int \frac{du}{u} = \ln|u| + c = \ln\left|\sin^2 x\right| + c$$

<u>Theorem</u>

1. $\int \tan x = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$

2. $\int \cot x = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$

3. $\int \sec x = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$

$$= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$

$$= \ln|\sec x + \tan x| + c$$

4. $\int \csc x = \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x}\,dx$

$$= \int \frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}\,dx$$

$$= -\int \frac{-\csc^2 x - \csc x \cot x}{\csc x + \cot x} dx$$

$$= -\ln\left|\csc x + \cot x\right| + c$$

Evaluate

1. $= \int \frac{\ln x}{x} dx = \int (\ln x) \frac{1}{x} dx = \frac{(\ln x)^2}{2} + c$

2. $= \int \frac{(\ln x)^3}{x} dx = \int (\ln x)^3 \frac{1}{x} dx = \frac{(\ln x)^4}{4} + c$

3. $= \int \frac{dx}{x \ln x} = \int \frac{(x)^{-1}}{\ln x} dx = \ln\left|\ln x\right| + c$

4. $= \int \frac{\sqrt{1 - \ln x}}{x} dx = \int (1 - \ln x)^{1/2} \frac{1}{x} dx$

$$= -\frac{(1 - \ln x)^{3/2}}{3/2} + c$$

<u>Exponential function</u>

1. If $y = e^u \quad \rightarrow y' = e^u \frac{du}{dx}$

$$\boxed{\int e^u du = e^u + c}$$

2. $y = a^u \quad \rightarrow y' = a^u \ln a \frac{du}{dx}$

$$\boxed{\int a^u du = \frac{a^u}{\ln a} + c}$$

EXAMPLE 1: $\int \frac{e^x}{1 + 3e^x} dx = \frac{1}{3} \int 3e^x (1 + 3e^x)^{-1} dx$

$$=\frac{1}{3}\int\frac{3e^x}{(1+3e^x)}dx$$

$$=\frac{1}{3}\ln\left|(1+3e^x)\right|+c$$

EXAMPLE 2: $\int 3x^3e^{-2x^4}dx=-\frac{3}{8}\int -8x^3\,e^{-2x^4}dx$

$$=-\frac{3}{8}e^{-2x^4}+c$$

EXAMPLE 3: $\int 2^{-4x}dx=-\frac{1}{4}\int 2^{-4x}(-4dx)$

$$=-\frac{1}{4}2^{-4x}\frac{1}{\ln 2}+c$$

EXAMPLE 4: $\int\frac{e^{\tan x}}{\cos^2 x}dx=\int e^{\tan x}\sec^2 x\,dx$

$=e^{\tan x}+c$

EXAMPLE 5: $\int\frac{e^x}{1+e^x}dx=\ln\left|1+e^x\right|+c$

EXAMPLE 6: $\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx=\ln\left|e^x+e^{-x}\right|+c$

EXAMPLE 7: $\int\frac{dx}{1+e^x}=\int\frac{dx}{1+e^x}\frac{e^{-x}}{e^{-x}}=\int\frac{e^{-x}\,dx}{1+e^{-x}}$

$$=\ln\left|1+e^{-x}\right|+c$$

EXAMPLE 6: $\int(\frac{e^{2x}-1}{e^{2x}+1})dx=\int(\frac{e^{2x}-1}{e^{2x}+1})\frac{e^{-x}}{e^{-x}}dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

$$=\ln\left|e^x+e^{-x}\right|+c$$

Invers function

1. **If** $y=\sin^{-1}u \quad \rightarrow \quad y'=\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

$$\boxed{\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}x+c}$$

$$\boxed{\int\frac{du}{\sqrt{a^2-u^2}}=\begin{cases}\sin^{-1}\frac{x}{a}+c\\-\cos^{-1}\frac{x}{a}+c\end{cases}}$$

2. $y=\tan^{-1}u \quad \rightarrow \quad y'=\dfrac{1}{1+u^2}\dfrac{du}{dx}$

$$\boxed{\int\frac{1}{1+u^2}du=\tan^{-1}x+c}$$

$$\boxed{\int\frac{1}{a^2+u^2}du=\begin{cases}\frac{1}{a}\tan^{-1}\frac{u}{a}\\-\frac{1}{a}\cot^{-1}\frac{u}{a}\end{cases}}$$

3. $y=\sec^{-1}u \quad \rightarrow \quad y'=\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$

$$\int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + c$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \begin{cases} \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| \\ -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| \end{cases}$$

EXAMPLES 1: $\int \frac{x\,dx}{\sqrt{1-x^4}}\,dx$

$u^2 = x^4$

$u = x^2$

$du = 2x\,dx$

Sol:

$$\frac{1}{2}\int \frac{2x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + c$$

$$= \frac{1}{2}\sin^{-1}x^2 + c$$

2: $\int \frac{x^2}{\sqrt{3-x^6}}\,dx$

$u^2 = x^6$

$u = x^3$

$du = 3x^2\,dx$

Sol:

$$= \frac{1}{3}\int \frac{3x^2}{\sqrt{3-x^6}}\,dx$$

$$= \frac{1}{3}\int \frac{du}{\sqrt{3-u^2}}$$

$$= \frac{1}{3}\sin^{-1}\frac{u}{\sqrt{3}} + c$$

$$= \frac{1}{3}\sin^{-1}\frac{x^3}{\sqrt{3}} + c$$

3: $\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x}{1+x^4}\,dx$

$$= \frac{1}{2}\tan^{-1}x^2 + c$$

4: $\int \frac{x}{4+x^4}\,dx = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}\frac{x^2}{2}\right) + c = \frac{1}{4}\tan^{-1}\frac{x^2}{2} + c$

5: $\int \frac{x^3}{7+x^8}\,dx$

$= \frac{1}{4}\int \frac{4x^3}{7+x^8}\,dx$

$= \frac{1}{4}\int \frac{du}{7+u^2}$

$= \frac{1}{4}\frac{1}{\sqrt{7}}\tan^{-1}\frac{u}{\sqrt{7}} + c = \frac{1}{4\sqrt{7}}\tan^{-1}\frac{x^4}{\sqrt{7}} + c$

$u^2 = x^8$
$u = x^4$
$du = 4x^3\,dx$

6: $\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx$

$= \int \frac{du}{1+u^2} = \tan^{-1} u + c$

$= \tan^{-1}(e^x) + c$

$u^2 = (e^x)^2$
$u = e^x$
$du = e^x\,dx$

7: $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx$

$= \sin^{-1}(e^x) + c$

8: $\int \frac{dx}{x(1+\ln x)} = \int \frac{x^{-1}dx}{(1+\ln x)}$

$= \int \frac{1/x\,dx}{(1+\ln x)}$

$= \ln(1+\ln x) + c$

9: $\int \frac{dx}{x(1+\ln x)^2} = \int (1+\ln x)^{-2}\,\frac{1}{x}dx$

$= \frac{(1+\ln x)^{-1}}{-1}$

10: $\int \frac{dx}{x(1+(\ln x)^2)} = \int \frac{1/x\,dx}{1+(\ln x)^2}$

$u^2 = (\ln x)^2$
$u = \ln x$
$du = \frac{1}{x}dx$

$$= \int \frac{du}{1+u^2} = \tan^{-1} u + c$$

$$= \tan^{-1}(\ln x)^2 + c$$

11: $$\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx = \sin^{-1}(\tan x) + c$$

12: $$\int \frac{2}{2x\sqrt{(2x)^2 - 1}}\, dx = \sec^{-1}(2x) + c$$

13: $$\int \frac{2}{\sqrt{x}(1+x)}\, dx = 4\int \frac{1/2\sqrt{x}\, dx}{1+(\sqrt{x})^2}$$

$$= 4\tan^{-1}\sqrt{x} + c$$

14: $$2\int \frac{\cos x\, dx}{1+(\sin x)^2} dx = 2\tan^{-1}(\sin x) + c$$

15: $$= \int \tan^{-1} x \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

16: $$= \int e^{\sin^{-1} x} \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

17: $$= \int \frac{\sin^{-1} x\, dx}{\sqrt{1-x^2}} = \frac{(\sin^{-1} x)^2}{2} + c$$

18: $$= \int \frac{\tan^{-1}(x)\, dx}{1+x^2} = \frac{(\tan^{-1}(x))^2}{2} + c$$

19: $$= \int e^{\tan^{-1} x} \frac{dx}{1+x^2} = e^{\tan^{-1} x} + c$$

20: $$= \int \frac{dx}{x\sec^{-1}(x)\sqrt{x^2-1}} = \int \frac{\dfrac{dx}{x\sqrt{x^2-1}}}{\sec^{-1}(x)}$$

$$= \ln\left|\sec^{-1} x\right| + c$$

21: $$\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \left(\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\right) dx$$

$$= \sin^{-1} x - \frac{1}{2}\int 2x(1-x^2)^{-1/2}\ dx$$

$$= \sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + c$$

22: $$\int \frac{1+x}{1+x^2}\,dx = \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right) dx$$

$$= \tan^{-1} x + \frac{1}{2}\ln\left|1+x^2\right| + c$$

23: $$\int e^{\ln(\tan^{-1} x)}\ \frac{dx}{1+x^2}$$

$$\int \tan^{-1} x\ \frac{dx}{1+x^2}$$

$$= \frac{(\tan^{-1} x)^2}{2} + c$$

<u>Hyperbolic Functions</u>

(1) If $\quad y = \sinh u \qquad y' = \cosh u \dfrac{du}{dx}$

$\qquad y = \cosh u \qquad y' = \sinh u \dfrac{du}{dx}$

$\qquad y = \tanh u \qquad y' = \operatorname{sech}^2 u \dfrac{du}{dx}$

1. $\int \sinh x\, dx = \cosh x + c$

2. $\int \cosh x\, dx = \sinh u + c$

3. $\int \operatorname{sech}^2 x\, dx = \tanh x + c$

اذا كانت مشتقة الاس متوفرة نستخدم القوانين التالية

(2) $\int \sinh^n ax \cosh ax\, dx = \dfrac{\sinh^{n+1}}{a(n+1)} + c$

(3) $\int \cosh^n ax \sinh ax\, dx = \dfrac{\cosh^{n+1}}{a(n+1)} + c$

(4) If $\int \sinh^n x\, dx$ or $\int \cosh^n x\, dx$

اذا كانت الدالة اسية والمشتقة غير متوفرة وكان الاس عدد زوجي

Case 1: if n is even, we use identity $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$, $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

اذا كانت الدالة اسية والمشتقة غير متوفرة وكان الاس عدد فردي

Case 2: if n is odd, we use identity $\cosh^2 x = \sinh^2 x + 1$, $\sinh^2 x = \cosh^2 x - 1$

<u>معكوس الدوال الزائدية</u>

(5) If $y = \sinh^{-1} u \quad \rightarrow \quad y' = \dfrac{1}{\sqrt{1+u^2}} \dfrac{du}{dx}$

$$\boxed{\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \tfrac{u}{a} + c}$$

$$\boxed{\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \tfrac{u}{a} + c}$$

(6) If $y = \tanh^{-1} u \quad \rightarrow y' = \frac{1}{1-u^2}\frac{du}{dx}$

$$\boxed{\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\tanh^{-1} u + c}$$

EXAMPLES: $\int \sinh^3 x\,dx$

$= \int \sinh^2 x \sinh x\, dx$

$= \int (\cosh^2 x - 1) \sinh x\, dx$

$= \int \cosh^2 x \sinh x\,dx - \int \sinh x\,dx$

$= \frac{\cosh^3 x}{3} - \cosh x + c$

***H.W* Ex 1:** $\int \cosh^4 2x\,dx$

***H.W* Ex 2:** $\int \frac{x\,dx}{1-x^4}$

***H.W* Ex 3:** $\int \frac{xdx}{\sqrt{x^4-1}}$

***H.W* Ex 4:** $\int \frac{x^2dx}{\sqrt{1+x^6}}$

EXAMPLE: $\int e^x \sinh 2x\,dx$

Sol:

$= \int e^x \left(\frac{e^{2x}-e^{-2x}}{2}\right)dx \qquad \sinh x = \frac{e^x - e^{-x}}{2}$

$= \int \frac{e^{3x}-e^{-x}}{2}\,dx$

$$=\frac{1}{2}\int e^{3x}\ dx-\frac{1}{2}\int e^{-x}\,dx$$

$$=\frac{1}{2}\int \frac{1}{3}e^{3x}\ 3dx+\frac{1}{2}\int e^{-x}\,(-dx)$$

$$=\frac{1}{2}[\frac{1}{3}e^{3x}+e^{-x}]+c$$

EXAMPLE: $\int \cosh(\ln\cos x)\,dx$

Sol:

$$=\int \frac{e^{\ln\cos x}+e^{-\ln\cos x}}{2}\ dx$$

$$=\frac{1}{2}\int e^{\ln\cos x}+e^{-\ln\cos x}$$

$$=\frac{1}{2}\int (\cos x+\frac{1}{\cos x})\,dx$$

$$=\frac{1}{2}\int (\cos x+\sec x)\,dx$$

$$=\frac{1}{2}\sin x+\frac{1}{2}\ln\left|\sec x+\tan x\right|+c$$

EXAMPLE: $\int \cosh(\ln x)\frac{dx}{x}=\sinh(\ln x)+c$

EXAMPLE: $\int \sinh(2x+1)\ dx=\frac{1}{2}\int \sinh(2x+1)\ 2dx$

$$=\frac{1}{2}\cosh(2x+1)+c$$

chapter (5) اسئلة متنوعة

<u>EXAMPLES:</u>

1. $\int (x^2-1)(4-x^2)\ dx=\int (4x^2-x^4-4+x^2)\ dx$

$$=\frac{5x^3}{3}-\frac{1}{5}x^5-4x+c$$

2. $\int e^x \sin e^x\ dx=-\cos e^x+c$

3. $\int \tan(3x+5)\,dx=\frac{1}{3}\int \tan(3x+5)\ 3dx=-\frac{1}{3}\ln\left|\cos(3x+5)\right|+c$

4. $\int \frac{\cot(\ln x)}{x}\,dx=\ln\left|\sin(\ln x)\right|+c$

5. $\int \frac{\sin x+\cos x}{\cos x}\,dx=\int \frac{\sin x}{\cos x}\,dx+\int \frac{\cos x}{\cos x}\,dx$

$$=-\ln\left|\cos x)\right|+x+c$$

6. $\int \frac{dx}{1-\cos x} dx$

Sol:

$$\int \frac{dx}{1-\cos x} \; \frac{1-\cos x}{1-\cos x} \;\Rightarrow\; \int \frac{1-\cos x}{1-\cos^2 x} dx$$

$$= \int \frac{1-\cos x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} dx - \int \frac{\cos x}{\sin^2 x} dx$$

$$= \int \csc^2 x \, dx - \int \frac{\cos x}{\sin x} \frac{1}{\sin x} dx = -\cot x + \csc x + c$$

7. $\int \cot(2x+1)\csc^2(2x+1)\, dx$

$$-\frac{1}{2}\int \cot(2x+1)(-2\csc^2(2x+1))\, dx$$

$$-\frac{1}{2}\frac{\cot^2(2x+1)}{2} + c = -\frac{1}{4}\cot^2(2x+1) + c$$

8. $\int \frac{dx}{\sqrt{1-9x^2}} dx$ $\qquad \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$

Sol:

$$\frac{1}{3}\int \frac{3\,dx}{\sqrt{1-(3x)^2}} dx = \frac{1}{3}\sin^{-1}(3x) + c$$

$u^2 = 9x^2$

$u = 3x$

$du = 3\,dx$

9. $\int \frac{dx}{\sqrt{2-x^2}} dx = \sin^{-1}\frac{x}{\sqrt{2}} + c$

10. $\int e^{2x} \cosh e^{2x}\, dx$

$u = e^{2x}$

$du = 2e^{2x}\, dx$

$$= \frac{1}{2}\int \cosh u \, du$$

$$=\frac{1}{2}\sinh u+c=\frac{1}{2}\sinh e^{2x}+c$$

11. $\int e^{\sin x}\cos x\,dx=e^{\sin x}+c$

12. $\int\frac{dx}{e^{3x}}dx=\int e^{-3x}dx=-\frac{1}{3}\int(-3e^{-3x})dx=-\frac{1}{3}e^{-3x}+c$

13. $\int\frac{e^{\sqrt{x}}-1}{\sqrt{x}}\ dx=\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx-\int\frac{dx}{\sqrt{x}}$

$$\frac{1}{2}\int\frac{e^{\sqrt{x}}}{2\sqrt{x}}dx-\int x^{-1/2}\,dx$$

$$=\frac{1}{2}e^{\sqrt{x}}+\frac{x^{1/2}}{1/2}+c=\frac{1}{2}e^{\sqrt{x}}+2\sqrt{x}+c$$

14. $\int x(a+b\sqrt{3x})\,dx=\int xa\,dx+\int xb\sqrt{3x}\ dx$

$$=\int xa\,dx+\int b\sqrt{3}\,x^{3/2}\ dx$$

$$=a\,\frac{x^2}{2}+\frac{b\sqrt{3}\,x^{5/2}}{5/2}+c$$

15. $\int\frac{dx}{-1-x^2}dx\quad\Rightarrow\quad\int\frac{-dx}{1+x^2}dx=-\int\frac{dx}{1+x^2}dx$

$$=-\tan^{-1}x+c$$

16. $\int\frac{\cos\theta\,d\theta}{1+\sin^2\theta}=\int\frac{1}{1+(\sin\theta)^2}\cos\theta\,d\theta$

$$\Rightarrow\int\frac{du}{1+u^2}=\tan^{-1}u+c$$

$$=\tan^{-1}(\sin\theta)+c$$

تكامل الدوال الزائدية

1. $\int \sinh x \, dx = \cosh x + c$

2. $\int \cosh x \, dx = \sinh x + c$

3. $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

4. $\int \operatorname{csch}^2 x \, dx = -\coth x + c$

5. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

6. $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + c$

تكامل معكوس الدوال الزائدية

1. $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c$

2. $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c$

3. $\int \frac{1}{1-x^2} \, dx = \begin{cases} \tanh^{-1} + c & \text{if} \quad |x| < 1 \\ \coth^{-1} + c & \text{if} \quad |x| > 1 \end{cases}$

$$= \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$$

4. $\int \frac{1}{x\sqrt{1-x^2}} \, dx = -\operatorname{sech}^{-1} x + c$

5. $\int \frac{1}{x\sqrt{1+x^2}} \, dx = -\operatorname{csch}^{-1} x + c$