Chapter six

Methods of integration

طرق التكامل

1. Integration by parts التكامل بالتجزئة

تبنى هذه الطريقة على قاعدة مشتقة حاصل ضرب دالتين

$$d(u\,v) = udv + vdu$$

$$udv = d(u\,v) - vdu$$

$$\boxed{\int udv = u\,v - \int vdu}$$

نلجا الى هذه الطريقة اذا لم نتمكن من الحل بالطرق السابقة وتمكنا من تجزئة السؤال الى جزئين احدهما قابل للتكامل والاخر قابل للاشتقاق. حيث يجب ان نحصل على $\int udv = u\,v - \int vdu$ الذي يفترض ان يكون ابسط من صيغة التكامل الاول في السؤال.

كيفية يتم اختيار $dv\ ,\ u$

1. **الحالة الاولى** : اذا كان السؤال يحتوي على

1. Ln
2. invers

والمشتقة غير متوفرة عندها نختار u $\begin{matrix}\text{ln}\\ \text{invers}\end{matrix}$ ثم نشتقها والباقي يكامل

Find the integration
Evaluate

EXAMPLE 1: $\int x \ln x \, dx$

$$u = \ln x \qquad du = \frac{1}{x}dx \qquad dv = x\,dx \qquad v = \int x\,dx = \frac{x^2}{2}$$

$$\int udv = u\,v - \int vdu$$

$$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + c$$

EXAMPLE 2: $\int \ln x\,dx$

$u = \ln x \qquad du = \frac{dx}{x} \qquad dv = dx \qquad v = x$

$$\int u dv = u\,v - \int v du$$

$$= x \ln x - \int x\ \frac{dx}{x}$$

$$= x \ln x - x + c$$

EXAMPLE 3: $\int x \cos x\,dx$

$u = x \qquad du = dx \qquad dv = \cos x dx \qquad v = \int \cos x\,dx = \sin x$

$$\int u dv = u\,v - \int v du$$

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx$$

$$= x \sin x + \cos x + c$$

EXAMPLE 4: $\int \sin^{-1} x\,dx$

$u = \sin^{-1} x \qquad du = \frac{dx}{\sqrt{1-x^2}} \qquad dv = dx \qquad v = x$

$$\int u dv = u\,v - \int v du$$

$$\int \sin^{-1} x\,dx = x \sin^{-1} x - \int x\ \frac{dx}{\sqrt{1-x^2}}$$

$$= x \sin^{-1} x - \int x\ (1-x^2)^{-1/2}\,dx$$

$$= x\sin^{-1}x + \frac{1}{2}\int -2x\ (1-x^2)^{-1/2}\,dx$$

$$= x\sin^{-1}x + \frac{1}{2}\,\frac{(1-x^2)^{1/2}}{1/2} + c$$

EXAMPLE 5: $\int \tan^{-1}x\,dx$

$u = \tan^{-1}x \qquad du = \dfrac{dx}{1+x^2} \qquad dv = dx \qquad v = x$

$$\int u dv = u\,v - \int v du$$

$$= x\tan^{-1}x - \int x\ \frac{dx}{1+x^2}$$

$$= x\tan^{-1}x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

2. اذا لم تحتوي الدالة على $\ln$ او دالة معكوسة نختارالدالة التي اذا تم اشتقاقها لعدة مرات الى ان تصل الى الصفر هي u اما الباقي فهي dv

EXAMPLE 1: $\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + c$

EXAMPLE 1: $\int x e^{x} dx$

$u = x \qquad du = dx \qquad dv = e^x\,dx \qquad v = e^x$

$$\int u dv = u\,v - \int v du$$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + c$$

H.W

1. $\int x^2 e^{-x} dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$

2. $\int x\sin^2 x\,dx$

3. طريقة الجدول: اذا لم تحتوي على $\ln$ و invers وهنالك دالة تحتاج الى عدد من الاشتقاقات لكي تصل الى الصفر

EXAMPLE 1: $\int x\cos x\,dx$

$$\int x\cos x\,dx = x\sin x + \cos x + c$$

الاشتقاق المتكرر	التكامل المتكرر
x	$\cos x$
1	$\sin x$
0	$-\cos x$

EXAMPLE 2: $\int x^3 e^x\,dx$

$$= x^3 e^x - 3x^2 e^x + 6x\,e^x - 6e^x + c$$

u	dv
x^3	e^x
$3x^2$	e^x
$6x$	e^x
6	e^x
0	e^x

EXAMPLE 3: $\int x^3 e^{2x}\,dx$

$$= \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{6}{8}x\,e^{2x} - \tfrac{6}{16}e^{x} + c$$

u	dv
x^3	e^{2x}
$3x^2$	$\frac{1}{2}e^{2x}$
$6x$	$\frac{1}{4}e^{2x}$
6	$\frac{1}{8}e^{2x}$
0	$\frac{1}{16}e^{2x}$

EXAMPLE 4: $\int xe^x\,dx$

$$= xe^x - e^x + c$$

u	dv
x	e^x
1	e^x
0	e^x

امثلة اخرى

EXAMPLE 1: $\int \sec^3 x\,dx = \int \sec^2 x\ \sec x dx$

$u = \sec x \qquad du = \sec x \tan x\,dx \qquad dv = \sec^2 x\,dx \qquad v = \tan x$

$\int u dv = u\,v - \int v du$

$\int \sec^3 x\,dx = \sec x \tan x - \int \tan x \sec x \tan x\,dx$

$= \sec x \tan x - \int \tan^2 x \sec x\,dx$

$= \sec x \tan x - \int (\sec^2 x - 1)\ \sec x\,dx$

$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$

$= \int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx - \int \sec x\,dx$

$= \int \sec^3 x\,dx + \int \sec^3 x\,dx = \sec x \tan x - \int \sec x\,dx$

$= 2\int \sec^3 x\,dx = \sec x \tan x - \ln|\sec x + \tan x| + c$

$\int \sec^3 x\,dx = \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + c$

EXAMPLE 2: $\int e^x \cos x\,dx$

$u = e^x \qquad du = e^x\,dx \qquad dv = \cos x\,dx \qquad v = \sin x$

$\int u dv = u\,v - \int v du$

$\int e^x \cos x\,dx = e^x \sin x - \int \sin x\, e^x\,dx$

To find $\int \sin x \, e^x \, dx$

$u = e^x \qquad du = e^x \, dx \qquad dv = \sin x \, dx \qquad v = -\cos x$

$$\int u \, dv = -e^x \cos x + \int \cos x \, e^x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x \, dx] + c$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx + c$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + c$$

$$\int e^x \cos x \, dx = \frac{e^x \sin x}{2} + \frac{e^x \cos x}{2} + c$$

$$= \frac{1}{2}(e^x \sin x + e^x \cos x) + c$$

EXAMPLE 3: $\int \frac{x}{\sqrt{x+1}} \, dx$

Sol:

u	dv
x	$(1+x)^{-1/2}$
1	$\frac{(1+x)^{1/2}}{1/2}$
0	$\frac{(1+x)^{3/2}}{(3/2)(1/2)}$

$$\int x(x+1)^{-1/2} \, dx$$

$$= 2x(x+1)^{1/2} - \tfrac{4}{3}(x+1)^{3/2} + c$$

u	dv
x	$(1-x)^{-1/2}$
1	$\frac{(1-x)^{1/2}}{1/2}$

0	$\frac{(1-x)^{3/2}}{(3/2)(1/2)}$

EXAMPLE 4: $\int \frac{x}{\sqrt{x-1}}\,dx = \int x(x-1)^{-1/2}\,dx$

$= 2x(x-1)^{1/2} - \frac{4}{3}(x-1)^{3/2}$

EXAMPLE 5: $\int x\sec^2 x\,dx$

$u = x \qquad du = dx \qquad dv = \sec^2 x\,dx \qquad v = \tan x$

$\int u\,dv = u\,v - \int v\,du$

$\int x\,\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$= x\tan x + \ln|\cos x| + c$

EXAMPLE6: $\int \sin^{-1} ax\,dx$

$u = \sin^{-1} ax \qquad du = \frac{a\,dx}{\sqrt{1-a^2x^2}} \qquad dv = dx \qquad v = x$

$\int u\,dv = u\,v - \int v\,du$

$\int \sin^{-1} ax\,dx = x\sin^{-1} x - \int x\frac{a\,dx}{\sqrt{1-a^2x^2}}$

$= x\sin^{-1} ax - \int a\,x\,(1-a^2x^2)^{-1/2}\,dx$

$= x\sin^{-1} ax + \frac{1}{2a}\frac{(1-a^2x^2)^{1/2}}{1/2} + c$

$$= x\sin^{-1} ax + \frac{1}{a}\,(1-a^2x^2)^{1/2} + c$$

EXAMPLE 7: $\int e^{ax} \sin bx \, dx$

$$u = e^{ax} \qquad du = a\,e^{ax}\,dx \qquad dv = \sin bx\,dx \qquad v = \frac{-\cos bx}{b}$$

$$\int u dv = u\,v - \int v du$$

$$\int e^{ax} \sin bx\; dx = e^{ax}(\frac{-\cos bx}{b}) - \int \frac{-\cos x}{b} a e^{ax}\, dx$$

$$\int e^{ax} \sin bx\; dx = e^{ax}(\frac{-\cos bx}{b}) + \frac{a}{b}\int \cos bx\, e^{ax} dx$$

$$u = e^{ax} \qquad du = a\,e^{ax}\,dx \qquad dv = \cos bx\,dx \qquad v = \frac{\sin bx}{b}$$

$$\int e^{ax} \cos bx\; dx = e^{ax}(\frac{\sin bx}{b}) - \int \frac{\sin bx}{b} a\; e^{ax} dx$$

$$\int e^{ax} \sin bx\; dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b}[\frac{1}{b}\; e^{ax} \sin bx - \frac{a}{b}\int \sin bx\, e^{ax}\; dx]$$

$$\int e^{ax} \sin bx\; dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2}\; e^{ax} \sin bx - \frac{a^2}{b^b}\int \sin bx\, e^{ax}\; dx$$

$$(1+\frac{a^2}{b^b})\int e^{ax} \sin bx\; dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2}\; e^{ax} \sin bx$$

$$\int e^{ax} \sin bx\; dx = -\frac{1}{b\;(1+\frac{a^2}{b^b})} e^{ax} \cos bx + \frac{a}{b^2}\frac{e^{ax} \sin bx}{(1+\frac{a^2}{b^b})} + c$$

EXAMPLE 8: $\int x^3\, e^{x^2}\, dx$

$$u = x^2 \qquad du = 2x\,dx \qquad dv = x\; e^{x^2}\; dx \qquad v = \tfrac{1}{2} e^{x^2}$$

$$\int u\,dv = u\,v - \int v\,du$$

$$\int x^2 x e^{x^2} = x^2\,\tfrac{1}{2}e^{x^2} - \int \tfrac{1}{2}e^{x^2}\,2x\,dx$$

$$= x^2\,\tfrac{1}{2}e^{x^2} - \tfrac{1}{2}e^{x^2} + c$$

EXAMPLE 9: $\int x^2 \cos x\,dx$

$u = x^2 \qquad du = 2x\,dx \qquad dv = \cos x\,dx \qquad v = \sin x$

$$\int u\,dv = u\,v - \int v\,du$$

$$\int x^2 \cos x\,dx = x^2 \sin x - \int \sin x\; 2x\,dx$$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\int \sin x\,x\,dx$$

$$\int x \sin x\;dx = uv - \int v\,du$$

$u = x \qquad du = dx \qquad dv = \sin x\,dx \qquad v = -\cos x$

$$\int x \sin x\,dx = -x\cos x - \int -\cos x\;dx$$

$$= -x\cos x + \sin x + c$$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2[-x\cos x + \sin x] + c$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + c$$

Or

$$= x^2 \sin x + 2x\cos x - 2\sin x + c$$

u	dv
x^2	$\cos x$
$2x$	$\sin x$
2	$-\cos x$
0	$-\sin x$

EXAMPLE 10: $\int \sin^{-1}(2x)\,dx$

$u = \sin^{-1}(2x) \qquad du = \dfrac{2\,dx}{\sqrt{1-(2x)^2}} \qquad dv = dx \qquad v = x$

$$\int u\,dv = u\,v - \int v\,du$$

$$\int \sin^{-1}(2x)\,dx = x\sin^{-1}(2x) - \int x\frac{2}{\sqrt{1-(2x)^2}}\,dx$$

$$= x\sin^{-1}(2x) - \int 2x(1-4x^2)^{-1/2}\,dx$$

$$= x\sin^{-1}(2x) + \frac{1}{4}\frac{(1-4x^2)^{1/2}}{1/2} + c$$

$$= x\sin^{-1}(2x) + \frac{1}{2}(1-4x^2)^{1/2} + c$$

(1) Trigonometric substitution integration التعويض باستخدام الدوال المثلثية

If we have

$a^2 - u^2$	Special case	$a=1$	$1-u^2$
$a^2 + u^2$	Special case	$a=1$	$1+u^2$
$u^2 - a^2$	Special case	$a=1$	u^2-1

Case 1 **في الحالة الاولى** شكل a^2-u^2 ياخذ الصور التالية

a. $a^2 - u^2$

b. $\sqrt{a^2 - u^2}$

c. $(a^2 - u^2)^n$

Let $u = a\sin\theta$ $\quad \frac{u}{a} = \sin\theta$ $\quad \theta = \sin^{-1}\frac{u}{a}$ $\quad du = a\cos\theta\,d\theta$

And used $1-\sin^2\theta = \cos^2\theta$

EXAMPLE: $\int \frac{x}{1-x^2}\,dx$ $\quad a^2-u^2$, $1-x^2$ $\quad a=1$, $u^2 = x^2$ $\quad \theta = \sin^{-1} x$

Let $x = a\sin\theta$, $x = \sin\theta$ $\quad dx = \cos\theta\,d\theta$

$$= \int \frac{\sin\theta\cos\theta}{1-\sin^2\theta}\, d\theta$$

$$= \int \frac{\sin\theta\cos\theta}{\cos^2\theta}\, d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta}\, d\theta$$

$$= -\ln\left|\cos\theta\right| + c$$

$$= -\ln\left|\cos(\sin^{-1} x)\right| + c$$

EXAMPLE: $\int \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int -2x\,(1-x^2)^{-1/2}\, dx$

$$= -\frac{1}{2}\,\frac{(1-x^2)^{1/2}}{1/2} + c$$

or

Let $x = a\sin\theta$, $x = \sin\theta$ $\qquad dx = \cos\theta\, d\theta \qquad \theta = \sin^{-1} x$

$$= \int \frac{\sin\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta$$

$$= \int \frac{\sin\theta\cos\theta}{\cos\theta}\, d\theta$$

$$= \int \sin\theta\, d\theta = \cos\theta + c$$

$$= \cos(\sin^{-1} x) + c$$

EXAMPLE: $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

Let $x = a\sin\theta$, $x = \sin\theta$ $\qquad dx = \cos\theta\, d\theta \qquad \theta = \sin^{-1} x$

$$=\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$$

$$=\int \frac{\cos\theta}{\cos\theta}\,d\theta$$

$$=\int d\theta=\theta+c$$

$$=\sin^{-1}x+c$$

Prove $\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c$

Let $x=a\sin\theta \Rightarrow \frac{x}{a}=\sin\theta$ $\quad dx=a\cos\theta\,d\theta$ $\quad \theta=\sin^{-1}\frac{x}{a}$

$$=\int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\,d\theta$$

$$=\int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\,d\theta$$

$$=\int \frac{a\cos\theta}{\sqrt{a^2\cos^2\theta)}}\,d\theta=\int d\theta=\theta+c$$

$$=\sin^{-1}\frac{x}{a}+c$$

EXAMPEL: $\int \sqrt{1-x^2}\,dx$

Let $x=\sin\theta$ $\quad dx=\cos\theta\,d\theta$ $\quad \theta=\sin^{-1}x$

$$=\int \sqrt{1-\sin^2\theta}\;\cos\theta\,d\theta$$

$$=\int \cos\theta\,\cos\theta\,d\theta$$

$$=\int \cos^2\theta \ d\theta$$

$$=\int (\frac{1}{2}+\frac{1}{2}\cos 2\theta)\ d\theta$$

$$=\frac{1}{2}\theta+\frac{1}{4}\ \sin 2\theta+c$$

$$=\frac{1}{2}\sin^{-1}x+\frac{1}{4}\ \sin 2(\sin^{-1}x)+c$$

EXAMPLE 1: $\int x\ \sin^{-1}x\,dx$

$$u=\sin^{-1}x \qquad du=\frac{dx}{\sqrt{1-x^2}} \qquad dv=x\,dx \qquad v=x^2/2$$

$$\int u dv=u\,v-\int v du$$

$$=\frac{x^2}{2}\sin^{-1}x-\int\frac{x^2}{2}\frac{dx}{\sqrt{1-x^2}}$$

$$=\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int x^2\frac{dx}{\sqrt{1-x^2}}$$

$$\int\frac{x^2\,dx}{\sqrt{1-x^2}} \qquad \Rightarrow x=a\sin\theta \quad a=1 \quad \sin^{-1}x=\theta$$

$$dx=\cos\theta\,d\theta$$

$$=\int\frac{\sin^2\theta\ \cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}}=\int\sin^2\theta\ d\theta$$

$$=\frac{1}{2}\theta-\frac{1}{4}\ \sin(2\sin^{-1}x)+c$$

$$\int x\ \sin^{-1}x\,dx=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{2}[\frac{1}{2}\sin^{-1}x-\frac{1}{4}\ \sin(2\sin^{-1}x)]+c$$

Case (2) **الحالة الثانية**

a. $u^2 - a^2$

b. $\sqrt{u^2 - a^2}$

c. $(u^2 - a^2)^n$

Let $u = a\sec\theta$ $\quad\theta = \sec^{-1}\frac{u}{a}$ $\quad du = a\sec\theta\,\tan\theta\,d\theta$

And used $\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 1: $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c$

(2) Let $x = a\sec\theta$ $\quad a = 1$

$x = \sec\theta$ $\quad\theta = \sec^{-1} x$ $\quad dx = \sec\theta\,\tan\theta\,d\theta$

$$\int \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta - 1}} = \int \sec\theta\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \ln\left|\sec(\sec^{-1} x) + \tan(\sec^{-1} x)\right| + c$$

EXAMPLE 2: $\int \frac{dt}{\sqrt{25t^2 - 9}} = \int \frac{dt}{\sqrt{(5t)^2 - (3)^2}}$

Let $u^2 = 25t^2$ $\quad a^2 = (3)^2$

$u = 5t = a\sec\theta$ $\quad a = 3$

$5t = 3\sec\theta$

$t = \frac{3}{5}\sec\theta$ $\quad \theta = \sec^{-1}\frac{5}{3}t$

$dt = \frac{3}{5}\sec\theta\ \tan\theta\ d\theta$

$$\int \frac{\frac{3}{5}\sec\theta\ \tan\theta\ d\theta}{\sqrt{9\sec^2\theta - 9}} = \int \frac{\frac{3}{5}\sec\theta\ \tan\theta\ d\theta}{3\sqrt{\sec^2\theta - 1}}$$

$$\int \frac{1}{5}\sec\theta\, d\theta = \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \frac{1}{5}\ln\left|\sec(\sec^{-1}(\tfrac{5}{3}t)) + \tan(\sec^{-1}(\tfrac{5}{3}t))\right| + c$$

Case (3) الحالة الثالثة

a. $a^2 + u^2$
b. $\sqrt{a^2 + u^2}$
c. $(a^2 + u^2)^n$

Let $u = a\tan\theta$ $\quad \theta = \tan^{-1}\frac{u}{a}$ $\quad du = a\sec^2\theta\, d\theta$

And used $\quad 1 + \tan^2\theta = \sec^2\theta$

EXAMPLE 1: $\int \frac{dx}{\sqrt{4 + x^2}} = \cosh^{-1}\frac{x}{2} + c$

(2) Let $\quad x = a\tan\theta$ $\quad a = 2$

$x = 2\tan\theta$ $\quad \theta = \tan^{-1}\frac{x}{2}$ $\quad dx = 2\sec^2\theta\ d\theta$

$$\int \frac{2\sec^2\theta\, d\theta}{\sqrt{4+4\tan^2\theta}} = \int \frac{2\sec^2\theta\, d\theta}{2\sqrt{1+\tan^2\theta}}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \ln\left|\sec(\tan^{-1}\tfrac{x}{2}) + \tan(\tan^{-1}\tfrac{x}{2})\right| + c$$

EXAMPLE 1: $\int \frac{z^2 dz}{\sqrt{1+z^2}}$

Let $z = a\tan\theta$ $\quad a = 1$

$z = \tan\theta$ $\quad \theta = \tan^{-1} z$ $\quad dz = \sec^2\theta\ d\theta$

$$\int \frac{\tan^2\theta \sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\tan^2\theta\ \sec^2\theta\, d\theta}{\sec\theta}$$

$$= \int \tan^2\theta\ \sec\theta\, d\theta = \int (\sec^2\theta - 1)\ \sec\theta\, d\theta$$

$$= \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$$

$$= \int \sec^3\theta\, d\theta - \ln\left|\sec\theta + \tan\theta\right| + c$$

$$\Rightarrow \int \sec^3\theta\, d\theta = \int \sec^2\theta \sec\theta\, d\theta$$

$u = \sec\theta$ $\quad du = \sec\theta\tan\theta\, d\theta$ $\quad dv = \sec^2\theta\, d\theta$ $\quad v = \tan\theta$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + c$$

$$\int \frac{z^2 dz}{\sqrt{1+z^2}} = \frac{1}{2}\sec(\tan^{-1} z)\tan(\tan^{-1} z) - \frac{1}{2}\ln\left|\sec(\tan^{-1} z) + \tan(\tan^{-1} z)\right|$$

$$-\ln\left|\sec(\tan^{-1} z) + \tan(\tan^{-1} z)\right| + c$$

EXAMPLE 2: $\int \frac{dy}{\sqrt{16+9y^2}}$

$a^2 = 16 \qquad a = 4$

$u^2 = 9y^2 \qquad u = 3y = a\tan\theta \qquad 3y = 4\tan\theta \qquad \frac{3}{4}y = \tan\theta$

$y = \frac{4}{3}\tan\theta \qquad \theta = \tan^{-1}(\frac{3}{4}y) \qquad dy = \frac{4}{3}\sec^2\theta \, d\theta$

$$\int \frac{\frac{4}{3}\sec^2\theta}{\sqrt{16+16\tan^2\theta}} \, d\theta = \int \frac{\frac{4}{3}\sec^2\theta}{4\sqrt{1+\tan^2\theta}} \, d\theta$$

$$\frac{1}{3}\int \sec\theta \, d\theta =$$

$$= \frac{1}{3}\ln\left|\sec(\tan^{-1}\frac{3}{4}y) + \tan(\tan^{-1}\frac{3}{4}y)\right| + c$$

3. <u>Partial fractions integration</u> — التكامل بالكسورالجزيئية

1. اولا

$$\int \frac{U^m(x)}{V^n(x)}dx \qquad \text{if} \;\; m \geq n$$

Used

$C(x)$

$V^n(x)$ | $U^m(x)$

a

$$\int \frac{U^m(x)}{V^n(x)}dx = \int (C(x) + \frac{a}{V(x)})\,dx$$

EXAMPLE 1:

$$\int \frac{x\,dx}{1+x}$$

1

$1+x$ | x

$\mp x \mp 1$

-1

$$\int (1 + \frac{-1}{1+x})\,dx$$

$$\int dx - \int \frac{dx}{1+x}$$

$$x - \ln|1+x| + c$$

EXAMPLE 2:

$$\int \frac{x^3}{x-1}\,dx$$

$$\int (x^2+x+1+\frac{1}{x-1})\,dx$$

$$\int x^2dx+\int x\,dx+\int dx+\int \frac{dx}{x-1}$$

$$\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x-1|+c$$

$$\begin{array}{r|l} & x^2+x+1 \\ \hline x-1 & x^3 \\ & \mp x^3 \pm x^2 \\ \hline & x^2 \\ & \mp x^2 \pm x \\ \hline & x \\ & \mp x \pm 1 \\ \hline & 1 \end{array}$$

EXAMPLE 3:

$$\int x\tan^{-1}x\,dx$$

$$u=\tan^{-1}x \quad \Rightarrow du=\frac{dx}{1+x^2}$$

$$dv=x\,dx \quad \Rightarrow v=\frac{x^2}{2}$$

$$\int u\,dv=uv-\int v\,du$$

$$=\int x\tan^{-1}x\,dx=\frac{x^2}{2}\tan^{-1}x-\int \frac{x^2}{2}\frac{dx}{1+x^2}$$

$$=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

$$=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int (1+\frac{-1}{1+x^2})\,dx$$

$$=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}(x-\tan^{-1}x)+c$$

$$\begin{array}{r|l} & 1 \\ \hline 1+x^2 & x^2 \\ & -x^2 \mp 1 \\ \hline & -1 \end{array}$$

EXAMPLE 4:

$$\int \frac{3x+2}{3x-1}\,dx$$

$$\int (1+\frac{3}{3x-1})\,dx$$

$$\int dx+\int \frac{3}{3x-1}dx$$

$$\begin{array}{r|l} & 1 \\ \hline 3x-1 & 3x+2 \\ & \mp 3x \pm 1 \\ \hline & 3 \end{array}$$

$$x + \ln|3x - 1| + c$$

<u>ثانيا</u>

$$\int \frac{U(x)}{V(x)} dx$$

If $V(x) = (x+a)(x+b(x+c)(x+E)$........

$$\frac{U(x)}{V(x)} = \frac{U(x)}{(x+a)(x+b)(x+c)(x+E)}$$

$$\frac{U(x)}{V(x)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)} + \frac{D}{(x+E)}$$

If $V(x) = (x^2+a)(x^2+b)(x^2+c)$.......

$$\frac{U(x)}{V(x)} = \frac{U(x)}{(x^2+a)(x^2+b)(x^2+c)} =$$

$$\frac{U(x)}{V(x)} = \frac{Ax+B}{(x^2+a)} + \frac{Cx+D}{(x^2+b)} + \frac{Ex+f}{(x^2+c)}$$

If $V(x) = (x+a)^n$

$$\frac{U(x)}{(x+a)^n} = \frac{A}{(x+a)} + \frac{B}{(x+a)} + \cdots\cdots\cdots + \frac{Z}{(x+a)^n}$$

And last step must be find the values of A,B,C------and etc.

Evaluate

1. $\int \frac{2x+9}{x^2-9}dx = \int \frac{2x+9}{(x-3)(x+3)}dx = \int (\frac{A}{(x-3)} + \frac{B}{(x+3)})\,dx$

$$\frac{2x+9}{x^2-9} = \frac{A}{(x-3)} + \frac{B}{(x+3)}$$

$$\frac{2x+9}{(x+3)(x-3)} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

$$2x+9 = Ax+3A+Bx-3B$$

$$2x+9 = Ax+Bx+3A-3B$$

$$2x = (A+B)x$$

$$9 = 3(A-B)$$

$$A+B = 2$$

$$A-B = 3$$

$$2A = 5$$

$$A = \tfrac{5}{2}$$

$$\tfrac{5}{2} + B = 2$$

$$B = 2 - \tfrac{5}{2}$$

$$B = -\tfrac{1}{2}$$

$$=\int (\frac{A}{(x-3)}+\frac{B}{(x+3)})\,dx$$

$$=\int (\frac{(5/2)}{(x-3)}+\frac{(-1/2)}{(x+3)})\,dx$$

$$=\frac{5}{2}\ln|(x-3)|-\frac{1}{2}\ln|(x+3)|+c$$

2. $\int \frac{dx}{x^4-1}=\int \frac{A}{(x-1)}\,dx+\int \frac{B}{(x+1)}\,dx+\int \frac{Cx+D}{x^2+1}\,dx$

$$(x^4-4)=(x-1)(x+1)(x^2+1)$$

$$\frac{1}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$\frac{1}{(x-1)(x+1)(x^2+1)}=\frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)((x-1)(x+1)}{(x-1)(x+1)(x^2+1)}$$

$$1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)$$

$$1=Ax^3+Ax+Ax^2+A+Bx^3-Bx-B-Bx^2+Cx^3-Cx+Dx^2-D$$

x^3 $\quad A+B+C=0$

x^2 $\quad A-B+D=0$

x $\quad A-B-C=0$

الثوابت $A-B-D=1$

نعوض في القيم التي تصفر المقام

From $\quad \dfrac{A}{x-1} \Rightarrow x=1$

$A(x+1)(x^2+1)=1 \qquad \Rightarrow \; A(1+1)(1+1)=1 \qquad 4A=1$

$A=\dfrac{1}{4}$

$\dfrac{B}{x+1} \Rightarrow x=-1$

$B(x-1)(x^2+1)=1$

$B(-1-1)(1+1)=1 \;\; \Rightarrow \quad B=-\dfrac{1}{4}$

نعوض A , B في المعادلات

$\dfrac{1}{4}-\dfrac{1}{4}+C=0$

$C=0$

And

$\dfrac{1}{4}-\dfrac{-1}{4}-D=1$

$D=\dfrac{1}{2}-1=-\dfrac{1}{2}$

$$\int \frac{dx}{x^4-1}dx=\int \frac{A}{(x-1)}\,dx+\int \frac{B}{(x+1)}dx+\int \frac{Cx+D}{x^2+1}\,dx$$

$$=\int \frac{1/4}{(x-1)}\,dx+\int \frac{-1/4}{(x+1)}dx+\int \frac{-1/2}{x^2+1}\,dx$$

$$=\int \frac{1/4}{(x-1)}\,dx+\int \frac{-1/4}{(x+1)}dx+\int \frac{-1/2}{x^2+1}\,dx$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\tan^{-1}x+c$$

طريقة (z)

تستخدم هذه الطريقة في حالة احتواء السؤال دوال مثلثية فقط

$$\text{Let } \sin x = \frac{2z}{1+z^2}$$

$$(1-z^2)^2 + (2z)^2 = (1+z^2)^2$$

$$1 - 2z^2 + z^4 + 4z^2 = 1 + 2z^2 + z^4$$

$$1 + 4z^2 - 2z^2 + z^4 = 1 + 2z^2 + z^4$$

$$1 + 2z^2 + z^4 = 1 + 2z^2 + z^4$$

$$\sin x = \frac{2z}{1+z^2} \qquad \cos x = \frac{1-z^2}{1+z^2} \qquad \sec x = \frac{1}{\cos x} \qquad \sec x = \frac{1+z^2}{1-z^2}$$

$$\csc x = \frac{1+z^2}{2z} \qquad \tan x = \frac{2z}{1-z^2} \qquad \cot x = \frac{1-z^2}{2z}$$

$$z = \tan \frac{x}{2}$$

$$\tan^{-1} z = \frac{x}{2}$$

$$dx = \frac{2\,dz}{1+z^2}$$

EXAMPLE 1: $$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}}$$

$$= \int \frac{2z/1+z^2}{\frac{1+z^2+2z}{1+z^2}}dz$$

$$= \int \frac{2\,dz}{1+z^2+2z} = \int \frac{2\,dz}{z^2+2z+1}$$

$$= \int \frac{2\,dz}{(z+1)(z+1)} = \int \frac{2\,dz}{(z+1)^2}$$

$$= \int \frac{2\,dz}{(z+1)(z+1)} = \int 2\,(z+1)^{-2}\,dz$$

$$= 2\int (z+1)^{-2}\,dz$$

$$2\frac{(z+1)^{-1}}{-1}+c$$

$$2\frac{(\tan\frac{x}{2}+1)^{-1}}{-1}+c$$

EXAMPLE 2: $$\int \frac{dx}{1+\sin x+\cos x} \qquad \sin x = \frac{2z}{1+z^2} \qquad \cos x = \frac{1-z^2}{1+z^2}$$

$$= \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}}$$

$$=\int \frac{2z/1+z^2}{\frac{1+z^2+2z+(1-z^2)}{(1+z^2)}}dz$$

$$=\int \frac{2z/1+z^2}{\frac{1+z^2+2z+1-z^2)}{(1+z^2)}}dz$$

$$=\int \frac{2\,dz}{2+2z} = \ln|2+2z|+c$$

$$=\ln\left|2+2\tan\tfrac{x}{2}\right|+c$$

EXAMPLE 3: $$\int \frac{dx}{\sin x+\tan x} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{2z}{1+z^2}+\frac{2z}{1-z^2}}$$

$$=\int \frac{\frac{2\,dz}{1+z^2}}{\frac{2z(1-z^2)+2z(1+z^2)}{(1+z^2)(1-z^2)}}$$

$$=\int \frac{\frac{2\,dz}{1+z^2}}{\frac{2z-2z^3+2z+2z^3}{(1+z^2)(1-z^2)}}$$

$$=\int \frac{2\,dz}{\frac{4z}{1-z^2}} = \int \frac{(1-z^2)}{2z}dz$$

$$=\int \frac{dz}{2z} - \int \frac{z^2}{2z}dz$$

$$= \frac{1}{2}\ln|z| - \frac{1}{2}\frac{z^2}{2} + c$$

$$= \frac{1}{2}\ln|\tan \tfrac{x}{2}| - \frac{1}{4}(\tan \tfrac{x}{2})^2 + c$$

EXAMPLE 4: $\displaystyle \int \frac{\cos x\,dx}{1-\cos x} = \int \frac{\frac{1-z^2}{1+z^2}\frac{2\,dz}{1+z^2}}{1-\frac{1-z^2}{1+z^2}}$

$$= \int \frac{\frac{2(1-z^2)\,dz}{(1+z^2)^2}}{\frac{1+z^2-1+z^2}{(1+z^2)}} = \int \frac{2(1-z^2)}{2z^2(1+z^2)}dz$$

$$= \int \frac{1-z^2}{z^2(1+z^2)}dz = \int (\frac{Az+B}{z^2} + \frac{Cz+d}{1+z^2})\,dz$$

<u>طريقة الفرضية</u>

اذا لم نستطع الحل بكل الطرق السابقة وكانت الدالة معقدة

EXAMPLE :

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$$

نفرض ان

$$y = \sqrt{x}$$
$$x = y^2$$
$$dx = 2y\,dy$$

$$= \int \frac{2y\,dy}{y\,(1+y)} = \int \frac{2dy}{1+y}$$

$$= 2\int \frac{dy}{1+y}$$

$$= 2\ln|1+y| + c$$

$$= 2\ln|1+\sqrt{x}| + c$$

EXAMPLE:

$$\int \frac{x\,dx}{1+\sqrt{x}}$$

نفرض ان

$$y = \sqrt{x}$$
$$x = y^2$$
$$dx = 2y\,dy$$

$$= \int \frac{y^2\,2y\,dy}{1+y}$$

$$= \int \frac{2y^3 dy}{1+y}$$

$$= \int (2y^2 - 2y + 2 + \frac{-2dy}{1+y})\,dy$$

$$= \frac{2y^3}{3} - \frac{2y^2}{2} + 2y - 2\ln|1+y| + c$$

$$\begin{array}{r|l} & 2y^2 - 2y + 2 \\ 1+y & 2y^3 \\ & \mp 2y^3 \mp 2y^2 \\ & -2y^2 \\ & \pm 2y^2 \pm 2y \\ & 2y \\ & \mp 2y \mp 2 \\ & -2 \end{array}$$

$$= \frac{2(\sqrt{x})^3}{3} - x + 2\sqrt{x} - 2\ln\left|1+\sqrt{x}\right| + c$$

EXAMPLE: $$\int \frac{-dy}{y^2+5y+4} = \int \frac{-dy}{(y-4)(y-1)}$$

$$= \int \left(\frac{A}{y-4} + \frac{B}{y+1}\right) dy$$

تكملة الحل واجب بيتي

EXAMPLE: $$\int \frac{xdx}{x^2+4x+3} = \int \frac{xdx}{(x+1)(x+3)} = \int \left(\frac{A}{x+1} + \frac{B}{x+3}\right) dx$$

تكملة الحل واجب بيتي

طريقة اكمال المربع

اذا كان السؤال **لايمكن تحليله** ax^2+bx+c معامل $x^2=1$

$a(x^2+\frac{b}{a}x+\frac{c}{a})$ نضيف ونطرح نصف $\frac{x}{2}$ أي ان $(\frac{b}{2a})^2$

EXAMPLE 1: $\int \frac{dx}{x^2+2x+2}$

$$\int \frac{dx}{x^2+2x+1-1+2} = \int \frac{dx}{(x+1)^2+1}$$

$$= \tan^{-1}(x+1)+c$$

EXAMPLE 2: $\int \frac{dx}{2x^2+2x+1} = \int \frac{dx}{2(x^2+x+\frac{1}{2})}$

$$= \int \frac{dx}{2(x^2+x+\frac{1}{4}-\frac{1}{4}+\frac{1}{2})}$$

$$= \frac{1}{2}\int \frac{dx}{x^2+x+\frac{1}{4}-\frac{1}{4}+\frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{1}{2}}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\frac{(x+\frac{1}{2})}{\frac{1}{\sqrt{2}}} + c \qquad\qquad = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

EXAMPLE 3: $\int \frac{dx}{\sqrt{x^2-2x-8}} = \int \frac{dx}{\sqrt{x^2-2x+1-1-8}}$

$$= \int \frac{dx}{\sqrt{x^2-2x+1-9}}$$

$$= \int \frac{dx}{\sqrt{(x-1)^2-(3)^2}}$$

$$= \cosh^{-1}(\frac{x-1}{3}) + c$$

EXAMPLE 4: $\int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{dx}{\sqrt{x^2+2x+1-1+2}}$

$$= \int \frac{dx}{\sqrt{x^2+2x+1 \quad -1+2}} = \int \frac{dx}{\sqrt{(x+1)^2+1}}$$

$$= \int \frac{du}{\sqrt{u^2+1}} = \sinh^1 u + c$$

$$= \sinh^1(x+1) + c$$

EXAMPLE 5: $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{-x^2+2x}} = \int \frac{dx}{\sqrt{-(x^2-2x)}}$

$$= \int \frac{dx}{\sqrt{-(x^2 - 2x + 1 - 1)}} = \int \frac{dx}{\sqrt{-[(x-1)^2 - 1]}}$$

$$= \int \frac{dx}{\sqrt{1 - (x-1)^2}} = \sin^{-1}(x-1) + c$$

EXAMPLE 6:

$$\int \frac{\cos x\,dx}{\sqrt{4 - \cos^2 x}} = \int \frac{\cos x\,dx}{\sqrt{4 - (1 - \sin^2 x)}}$$

$$\int \frac{\cos x\,dx}{\sqrt{3 + \sin^2 x}} = \sinh^{-1}\left(\frac{\sin x}{\sqrt{3}}\right) + c$$

$u^2 = \sin^2 x$

$u = \sin x$

$du = \cos x\,dx$