Chapter seven

(المساحة) The Area

من التطبيقات المهمة للتكامل المحدد هو ايجاد المساحة تحت منحني الدالة $y=f(x)$ حيث ان $f(x)$ دالة مستمرة في الفترة $[a,b]$

1. المساحة المحددة بمنحني الدالة $y=f(x)$ ومحور السينات x-axis لايجاد ذلك نتبع الخطوات التالية.

نقاطع المنحني مع محور السينات x-axis وذلك بجعل $y=0$ لمعرفة $f(x)>0$ او $f(x)<0$

عندما $f(x)>0$ المساحة تساوي $A=\int_a^b f(x)\,dx$

وعندما $f(x)<0$ فان المساحة تستوي $A=-\int_a^b f(x)\,dx$

EXAMPLE: finds the area bounded by the x-axis and the curve $y=2x-x^2$

Sol:

$$y=2x-x^2$$

$$y=0$$

$$2x-x^2=0$$

$$x(2-x)=0$$

Either $x=0$ or $2-x=0$ $x=2$

$$\int_0^2 (2x-x^2)\,dx$$

$$\left.\frac{2x^2}{2}\right|_0^2-\left.\frac{x^3}{3}\right|_0^2$$

$$\left.x^2\right|_0^2-\left.\frac{x^3}{3}\right|_0^2=(4-0)-(\frac{8}{3}-0)$$

$$4-\frac{8}{3}=\frac{4}{3}$$

EXAMPLE: finds the area bounded by the y-axis and the curve $x = y^2 - y^3$

Sol:

لاحظ ان في السؤال مطلوب المساحة تحت المنحني مع المحور y y-axis هنا نجعل $x = 0$

$$x = y^2 - y^3$$

$$y^2 - y^3 = 0$$

$$y^2(1 - y) = 0$$

$$y^2 = 0 \qquad y = 0 \qquad \text{or} \qquad 1 - y = 0 \qquad y = 1$$

$$A = \int_0^1 x\,dy$$

$$A = \int_0^1 (y^2 - y^3)\,dy$$

$$\left.\frac{y^3}{3}\right|_0^2 - \left.\frac{y^4}{4}\right|_0^2$$

$$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

2. المساحة المحصورة بين دالتين

نجد نقاط التقاطع لايجاد حدود التكامل

$$A = \int_a^b (y_2 - y_x)\, dx \qquad \text{if} \quad y_2 > y_1$$

$$A = \int_a^b (y_1 - y_2)\, dx \qquad \text{if} \quad y_1 > y_2$$

EXAMPLE: finds the area bounded by the curve $y = x^2$ and the line $y = x$

Sol:

$$y = x^2$$

$$y = x$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

Either $x = 0$ or $x - 1 = 0$ $x = 1$

لاحظ خلال الفترة $[0, 1]$ ان $x > x^2$

$$\int_a^b (y_2 - y_1)\, dx = \int_0^1 (x - x^2)\, dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = [\frac{1}{2} - \frac{1}{3}] - [0 - 0]$$

$$= \frac{1}{6}$$

$$A = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} dy\, dx$$

$$A=\int_0^1 y\Big|_{x^2}^{x}\,dx=\int_0^1 (x-x^2)\,dx$$

$$\frac{x^2}{2}-\frac{x^3}{3}\Big|_0^1=\frac{1}{6}$$

EXAMPLE: finds the area bounded by the curve $y=x^2$ and the line $y=x+2$

Sol:

$$x^2-x-2=0$$

$$(x-2)(x+1)=0$$

$x=2$ or $x=-1$

لاحظ خلال الفترة $[-1, 2]$ ان $x+2>x^2$

$$A==\int_{-1}^{2}[(x+2)-x^2]\,dx$$

$$=\frac{x^2}{2}+2x-\frac{x^3}{3}\Big|_{-1}^{2}=$$

$$=[\frac{2^2}{2}+2\cdot 2-\frac{2^3}{3}]-[\frac{-1^2}{2}+2\cdot(-1)-\frac{(-1)^3}{3}]$$

$$=3$$

$$A=\int_{x=-1}^{x=2}\int_{y=x^2}^{y=x+2} dy\,dx$$

$$A=\int_{x=-1}^{x=2} y\int_{y=x^2}^{y=x+2}\quad dx=\int_{-1}^{2}(x+2)-(x^2)\,dx$$

$$=\frac{x^2}{2}+2x-\frac{x^3}{3}\Big|_{-1}^{2}=$$

$$=[\frac{2^2}{2}+2\cdot 2-\frac{2^3}{3}]-[\frac{-1^2}{2}+2\cdot(-1)-\frac{(-1)^3}{3}]$$

$$=3$$

EXAMPLE: Find the area of the triangular region in the first quadrant bounded by the y-axis and the curve $y=\sin x, y=\cos x$

Sol:

$$A=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)\,dx$$

$$=\sin x-(-\cos x)\Big|_0^{\frac{\pi}{4}}$$

$$=(\sin\tfrac{\pi}{4}+\cos\tfrac{\pi}{4})-(\sin 0+\cos 0)$$

$$=\frac{2}{\sqrt{2}}-1$$

$$A=\int_{x=0}^{x=\frac{\pi}{4}}\int_{y=\sin x}^{y=\cos x} dy\,dx=\int_0^{\frac{\pi}{4}} y\Big|_{\sin x}^{\cos x} dx$$

$$A=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)\,dx$$

$$=\sin x-(-\cos x)\Big|_0^{\frac{\pi}{4}}$$

$$=(\sin\tfrac{\pi}{4}+\cos\tfrac{\pi}{4})-(\sin 0+\cos 0)$$

$$=\frac{2}{\sqrt{2}}-1$$

EXAMPLE: finds the area bounded by the curve $y=x^4-2x^2$ and $y=2x^2$

Sol:

$$x^4 - 2x^2 - 2x^2 = 0$$

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x = 0 \qquad x^2 = 4 \qquad x = \pm 2$$

$$2\int_0^2 2x^2 - [x^4 - 2x^2]\,dx$$

$$2\int_0^2 (2x^2 - x^4 + 2x^2)\,dx$$

$$2\int_0^2 (4x^2 - x^4)\,dx$$

$$= 2\left(\frac{4x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^2$$

EX1: finds the area bounded by the curve $y = x^3$ and the line $y = x$

EX2: finds the area bounded by the curve $y = \sqrt{x}$ and the line $y = x$

EXAMPLE: Calculate $\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$

Sol:

نغير حدود التكامل

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x}\frac{\sin x}{x}\,dy\,dx=\int_{0}^{1}\frac{\sin x}{x}\,y\Big|_{0}^{x}\,dx$$

$$=\int_{0}^{1}\frac{\sin x}{x}\,x\,dx$$

$$=\int_{0}^{1}\sin x\,dx$$

$$-\cos x\Big|_{0}^{1}$$

$$=-(\cos 1-\cos 0)$$

$$=1.523048\times 10^{-4}$$