IN ELASTIC BUCKLING

INTROUDACTION

In the lectures presented heretofore, the assumption has been made that the material obeys Hooke's law. For this assumption to be valid, the stresses in the column must be below the proportional limit of the material. The linear elastic analysis is correct for slender columns. On the other hand, the axial stress in a shot column will exceed the proportional limit. Consequently, the elastic analysis is not valid for short columns, and the limiting load for short columns must be determined by taking inelastic behavior into account.

DOUBLE-MODULUS (REDUCED MODULUS) THEORY

Assumptions

1) Small displacement theory holds.
2) Plane sections remain plane. This assumption is called Bernoulli, or Euler, or Navier hypothesis.
3) The relationship between the stress and strain in any longitudinal fiber is given by the stress-strain diagram of the material (compression and tension, the same relationship).
4) The column section is at least singly symmetric, and the plane of bending is the plane of symmetry.
5) The axial load remains constant as the member moves from the straight to the deformed position.

In small displacement theory, the curvature of the bent column is

$$\frac{1}{R} \doteq \frac{d^2y}{dx^2} = \frac{d\phi}{dx}$$

From a similar triangle relationship, the flexural strains are computed

$$\varepsilon_1 = z_1 y''$$

$$\varepsilon_2 = z_2 y''$$

and the corresponding stresses are

$$\sigma_1 = E h_1 y''$$

$$\sigma_2 = E_t h_2 y''$$

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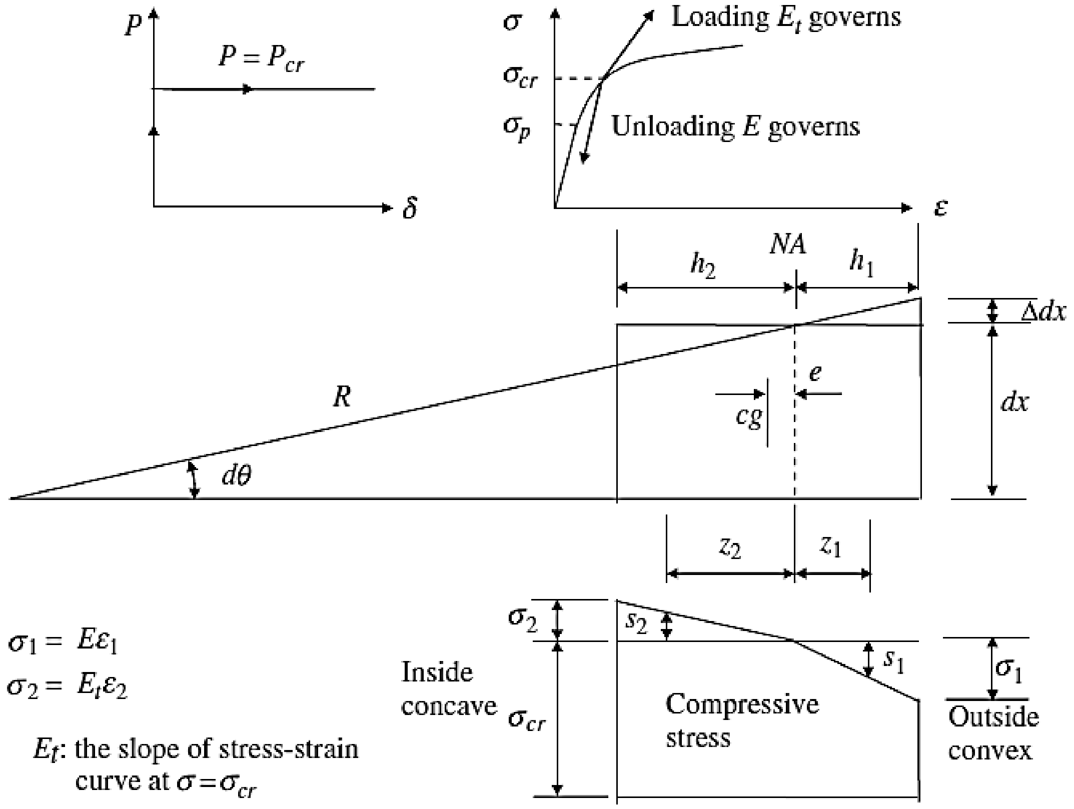

Figure 1: Reduced modulus model

where E_t = tangent modulus, s_1 (tension) $= Ez_1y''$ and s_2 (compression) $= E_tz_2y''$.

The pure bending portion (no net axial force) requires

$$\int_0^{h_1} s_1 dA + \int_0^{h_2} s_2 dA = 0$$

Equating the internal moment to the external moment yields

$$\int_0^{h_1} s_1 z_1 dA + \int_0^{h_2} s_2 z_2 dA = Py$$

$$Ey'' \int_0^{h_1} z_1 dA + E_t y'' \int_0^{h_2} z_2 dA = 0$$

Let $Q_1 = \int_0^{h_1} z_1 dA$ and $Q_2 = \int_0^{h_2} z_2 dA \Rightarrow EQ_1 + E_tQ_2 = 0$

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$$y''\left(E\int_0^{h_1} z_1^2 dA + E_t\int_0^{h_2} z_2^2 dA\right) = Py$$

Let $\overline{E} = \dfrac{EI_1 + E_tI_2}{I}$

Which is called the reduced modulus that depends on the stress-strain relationship of the material and the shape of the cross section. I_1 is the moment of inertia of the tension side cross section about the neutral axis and I_2 is the moment of inertia of the compression side cross section such that:

$$I_1 = \int_0^{h_1} z_1^2 dA \quad \text{and} \quad I_2 = \int_0^{h_2} z_2^2 dA$$

$$\overline{E}Iy'' + Py = 0 \qquad P_{r,cr} = \frac{\pi^2\overline{E}I}{\ell^2} \qquad \sigma_{r,cr} = \frac{\pi^2\overline{E}}{\left(\frac{\ell}{r}\right)^2}$$

Introducing

$$\tau_r = \frac{\overline{E}}{E} = \frac{E_t}{E}\frac{I_2}{I} + \frac{I_1}{I} < 1.0 \quad \text{and} \quad \tau = \frac{E_t}{E} < 1.0$$

the differential equation based on the reduced modulus becomes

$$EI\tau_r y'' + Py = 0$$

and

$$\tau_r = \tau\frac{I_2}{I} + \frac{I_1}{I} \quad \text{and} \quad \sigma_{r,cr} = \frac{P_{r,cr}}{A} = \frac{\pi^2 E\tau_r}{\left(\frac{\ell}{r}\right)^2}$$

The procedure for determining $\sigma_{r,cr}$ may be summarized as follows:

1) For $\sigma - \varepsilon$ diagram, prepare $\sigma - \tau$ diagram.
2) From the result of step 1, prepare $\tau_r - \sigma$ curve.
3) From the result of step 2, prepare $\sigma_r - (\ell/r)$ curve.

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TANGENT-MODULUS THEORY

Assumptions

The assumptions are the same as those used in the double-modulus theory, except assumption 5. The axial load increases during the transition from the straight to slightly bent position, such that the increase in average stress in compression is greater than the decrease in stress due to bending at the extreme fiber on the convex side. The compressive stress increases at all points; the tangent modulus governs the entire cross section. If the load increment is assumed to be negligibly small such that

$$\Delta P <<< P$$

then

$$E_t I y'' + P y = 0$$

and the corresponding critical stress is

$$\sigma_{t,cr} = \frac{P_{t,cr}}{A} = \frac{\pi^2 : E\tau}{\left(\frac{\ell}{r}\right)^2} \quad \text{with } \tau = \frac{E_t}{E}$$

Hence, σ_t vs ℓ/r curve is not affected by the shape of the cross section. The procedure for determining the $\sigma_t - (\ell/r)$ curve may be summarized as follows:

1) From $\sigma - \varepsilon$ diagram, establish $\sigma - \tau$ curve.
2) From the result of step 1, prepare $\sigma_t - (\ell/r)$.

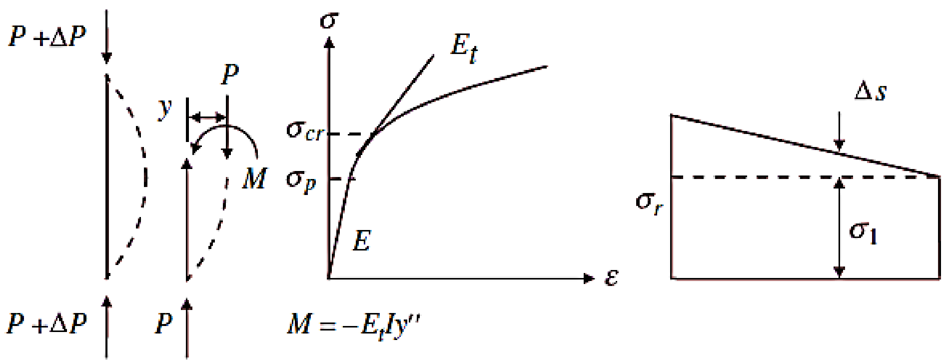

Figure 2: Tangent-modulus model

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ULTIMATE STRENGTH OF BEAM-COLUMNS

Up to this point in the study of beam-columns, the subject of failure was not considered; hence, it was possible to limit the discussion to elastic behavior. It is the modern trend that the design specifications are developed using the probability-based load and resistance factor design concepts: The load carrying capacity of each structural member all the way up to its ultimate strength needs to be evaluated. Since the ultimate strength of a structural member frequently involves yielding, it becomes necessary to work with the complexities of inelastic behavior in the analysis.

Consider the simply supported, symmetrically loaded beam-column shown in Fig. 3. demonstrated that a closed form solution for the load-deflection behavior beyond the proportional limit can be obtained when the following assumptions are made:

1. The cross section of the member is rectangular as shown in Fig. 3.
2. The material obeys linearly elastic and perfectly plastic stress-strain relationships.
3. The bending deflection of the member takes the form of a half-sine wave.

Inelastic bending is difficult to analyze because the stress-strain relation varies in a complex manner both along the member and across the section once the proportional limit has been exceeded. In addition to these major idealizations, which simplify the analysis greatly, the following assumptions are also made:

4. Deformations are finite but still small enough so that the curvature can be approximated by the second derivative of the deflected curve.
5. The member is initially straight.
6. Bending takes place about the major principal axis.

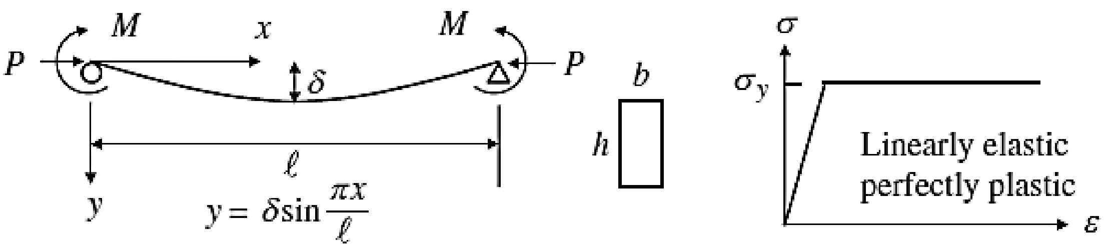

Figure 3: Idealized beam-column

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The residual stress that cannot be avoided in rolled and/or fabricated metal sections is ignored in the analysis.

Based on the coordinate system shown in Fig. 3, the external bending moment at a distance x from the origin is

$$M_{ext} = M + Py$$

Since this Eq. is an external equilibrium equation, it is valid regardless of whether or not the elastic limit of the material is exceeded. The relationship between the load and deflection up to the proportional limit is

$$M + Py = -EIy'' = -EI\left(-\delta\frac{\pi^2}{\ell^2}\right)\sin\frac{\pi x}{\ell}$$

or

$$M + Py = EI\frac{\delta\pi^2}{\ell^2}\sin\frac{\pi x}{\ell}$$

This relationship at the midspan becomes

$$M + P\delta = \frac{EI\delta\pi^2}{\ell^2}$$

Assuming that M is proportional to P, one introduces the notation $e = M/P$; then the above moment equation becomes

$$P(e+\delta) = \frac{\delta EI\pi^2}{\ell^2} = \delta P_E$$

Dividing both sides of this Eq. by h yields:

$$P\left(\frac{e}{h}+\frac{\delta}{h}\right) = \frac{\delta}{h}P_E$$

Where: $\sigma_E = P_E/bh$ is the Euler stress and $\sigma_0 = P/bh$ is the average axial stress.

This Equation gives the load versus deflection in elastic range. In the order to determine at which this Equation become invalid, one must evaluate the max. stress in the member.

$$\frac{\delta}{h} = \frac{e}{h}\frac{1}{\dfrac{\sigma_E}{\sigma_0}-1}$$

$$\sigma_{max} = \frac{P}{bh} + \frac{M + P\delta}{\dfrac{bh^2}{6}} = \sigma_0 + \sigma_0\frac{6(e+\delta)}{h}$$

$$\sigma_{max} = \sigma_0\left[1 + \frac{6(e+\delta)}{h}\right]$$

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As yielding propagates inward, the inner elastic core as indicated in

Fig. 4 for the inelastic case.

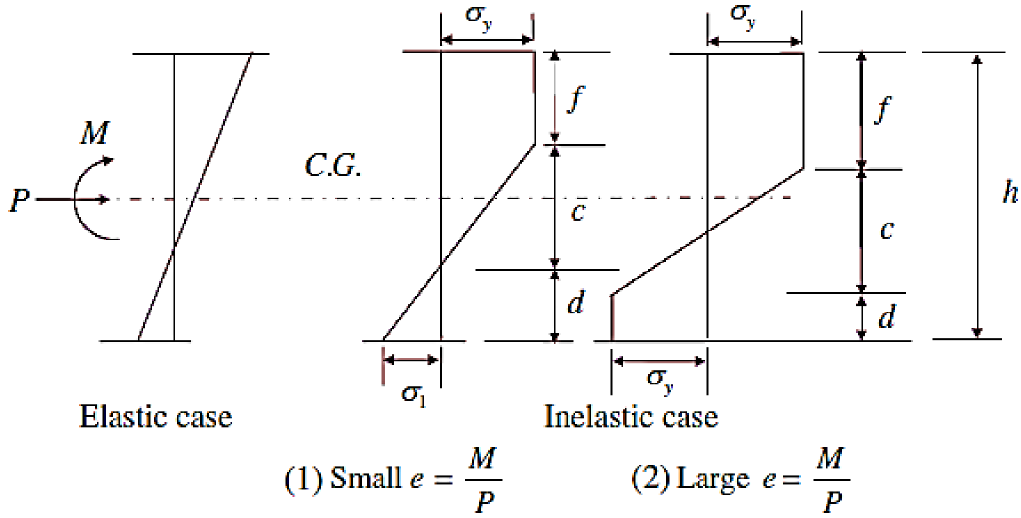

Figure 4: Stress distributions for beam-column

Depending on the eccentricity, two different stress distributions are possible. If the ratio $e=M/P$ is relatively small, yielding occurs only on the concave side of the member prior to reaching its ultimate strength range. On the other hand, if the eccentricityis relatively large, both the convex and concave sides of the member will have started to yield before the maximum load is reached, as shown in Fig. 4.
Summing the horizontal forces in case (1) of Fig. 4 yields:

$$P = b\left(\sigma_y f + \frac{\sigma_y c}{2} - \frac{\sigma_1 d}{2}\right)$$

Dividing both sides by bh yields

$$\sigma_0 = \frac{1}{h}\left(\sigma_y f + \frac{\sigma_y c}{2} - \frac{\sigma_1 d}{2}\right)$$

Summing the moment about the centroidal axis gives

$$M_{int} = \left[\sigma_y f\left(\frac{h}{2} - \frac{f}{2}\right) + \frac{\sigma_y c}{2}\left(\frac{h}{2} - f - \frac{c}{3}\right) + \frac{\sigma_1 d}{2}\left(\frac{h}{2} - \frac{d}{3}\right)\right]$$

Noting that $f + c + d = h$, c value can be determined from Eqs. above. After some lengthy algebraic manipulations, one obtains

$$c = \frac{9\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_0}-1\right)-\frac{M_{int}}{P}\right]^2}{2\sigma_0 h\left(\frac{\sigma_y}{\sigma_0}-1\right)^3}$$

$$\frac{\rho}{dx} = \frac{c}{\varepsilon_y dx}$$

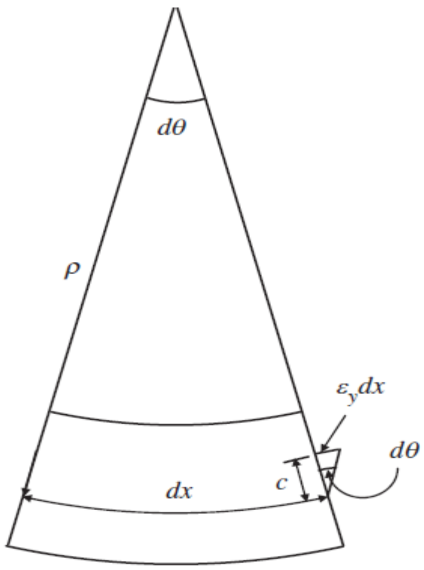

Figure 5: Similar triangle relationship

where ρ is the radius of curvature.
Thus

$$\frac{1}{\rho} = \frac{\varepsilon_y}{c} \doteq \frac{d^2y}{dx^2}$$

or

$$y'' = \frac{\sigma_y}{cE}$$

$$y'' = \frac{2\sigma_0 h\left(\frac{\sigma_y}{\sigma_0}-1\right)^3}{9E\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_0}-1\right)-(e+\delta)\right]^2}$$

The curvature and moment at mid-span are given by:

$$y''\Big|_{\ell/2} = \delta\frac{\pi^2}{\ell^2} \quad \text{from } y = \delta\sin\frac{\pi x}{\ell} \quad \text{and} \quad M_{int} = P(e+\delta)$$

$$\delta\frac{\pi^2}{\ell^2} = \frac{2\sigma_0 h\left(\frac{\sigma_y}{\sigma_0}-1\right)^3}{9E\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_0}-1\right)-(e+\delta)\right]^2} \text{ or}$$

$$\delta\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_0}-1\right)-e-\delta\right]^2=\frac{2h\ell^2\sigma_0}{9E\pi^2}\left(\frac{\sigma_y}{\sigma_0}-1\right)^3 \text{ or}$$

$$\frac{\delta}{h}\left[\frac{1}{2}\left(\frac{\sigma_y}{\sigma_0}-1\right)-\frac{e}{h}-\frac{\delta}{h}\right]^2=\frac{2\ell^2\sigma_0}{9E\pi^2h^2}\left(\frac{\sigma_y}{\sigma_0}-1\right)^3$$

Since $\sigma_E = \pi^2EI/(Al^2) = \pi^2Eh^2/(12l^2)$, the Eq. above can be rewritten in the form:

$$\frac{\delta}{h}\left[\frac{1}{2}\left(\frac{\sigma_y}{\sigma_0}-1\right)-\frac{e}{h}-\frac{\delta}{h}\right]^2=\frac{1}{54}\,\frac{\sigma_0}{\sigma_E}\left(\frac{\sigma_y}{\sigma_0}-1\right)^3$$

This Equation gives the load versus deflection relationship in the inelastic range.

H.W. A numerical application about double-modulus (reduced modulus) theory and tangent-modulus theory.