Ministry of Higher Education
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College of Science
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Theorem (1): (Properties of vector arithmetic):

(Algebraic properties of vectors)

Let **u**, **v**, and **w** are vectors in 2-or 3-space and *k* and *c* are scalars (real numbers), then the following relationships holds.

A) Addition properties:

a) **u** + **v** = **v** + **u** (**Commutative**).

b) (**u** + **v**) + **w** = **u** + (**v** + **w**) (**Associative**).

c) **u** + 0 = 0 + **u** = **u** (**Additive identity**).

d) **u** + (-**u**) = 0 (**Additive inverse**).

B) Scalar multiplication properties:

e) $k(c\mathbf{u}) = (kc)\mathbf{u}$ (**Associative property**).

f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (**Distributive property**).

g) $(k + c)\mathbf{u} = k\mathbf{u} + c\mathbf{u}$ (**Distributive property**).

h) $1\mathbf{u} = \mathbf{u}$ (**Multiplicative identity**).

Proof of part (b): (**u** + **v**) + **w** = **u** + (**v** + **w**). (**Analytic**).

If **u**, **v**, and **w** are three vectors in 3-space such that $\mathbf{u}=\langle u_1, u_2, u_3\rangle$, $\mathbf{v}=\langle v_1, v_2, v_3\rangle$ and $\mathbf{w}=\langle w_1, w_2, w_3\rangle$, then:

$$
\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \left[\langle u_1, u_2, u_3\rangle + \langle v_1, v_2, v_3\rangle\right] + \langle w_1, w_2, w_3\rangle \\
&= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle + \langle w_1, w_2, w_3\rangle \\
&= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3\rangle \\
&= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3)\rangle \\
&= \langle u_1, u_2, u_3\rangle + \langle (v_1 + w_1), (v_2 + w_2), (v_3 + w_3)\rangle \\
&= \mathbf{u} + (\mathbf{v} + \mathbf{w}).
\end{aligned}
$$

Similarly, for the proof in 2-space.

Now we shall proof this part **geometrically**.

If the vectors **u**, **v**, and **w** are represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$ as shown in **(figure 19),** then:

v + **w** = $\overrightarrow{QS}$ and **u** + (**v** + **w**) = $\overrightarrow{PS}$, (**Vector addition**).

u + **v** = $\overrightarrow{PR}$ and (**u** + **v**) + **w** = $\overrightarrow{PS}$, (**Vector addition**).

Thus, **u** + (**v** + **w**) = (**u** + **v**) + **w**.

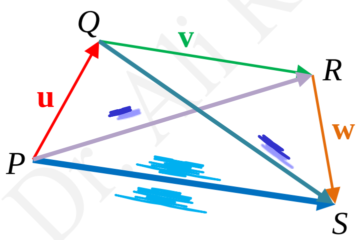

Figure 19

The vectors **u** + (**v** + **w**) and (**u** + **v**) + **w** are **equal.**

In **figure 19**, we note that the symbol **u** + **v** + **w** is clear since the same sum is obtained no matter where parentheses are inserted and if the vectors **u**, **v** and **w** are placed "tail-to-tip" then the sum **u** + **v** + **w** is the vector from the initial point of **u** to the terminal point of **w.**

Magnitude (length) or norm of a vector in 3-space:

How to visualize the norm geometrically in 3-space?

Previously we defined the magnitude or norm of the vector v in 2-space that is denoted by ||**v**|| and we said the same method would be to define the magnitude of the vector in 3-space and for higher- dimensional vector space.

If **v** = $<v_1, v_2, v_3>$ is any vector in 3-space as shown in **(figure 20)**. Using two application of the theorem of **Pythagoras**, we obtain:

$\|\mathbf{v}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (OS)^2 + (RP)^2 = v_1^2 + v_2^2 + v_3^2$

$\|\mathbf{v}\| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}.$

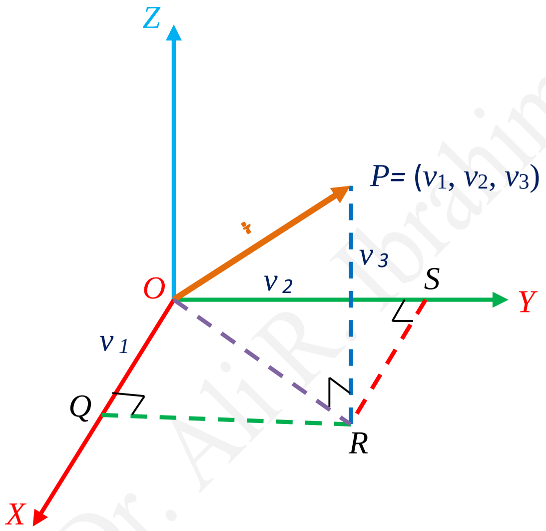

Figure 20

We know if the magnitude (length) or norm of the vector equal 1, then the vector is called a unit vector.

<u>Note:</u> If $\mathbf{v}$ is any nonzero vector, then for any scalar k, $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\| = k\|\mathbf{v}\|$.

Let $\mathbf{v} = \langle v_1, v_2 \rangle$ is any vector in 2-space, then:

$k\mathbf{v} = k\langle v_1, v_2\rangle = \langle kv_1, kv_2\rangle;$

Thus, $\|k\mathbf{v}\| = \sqrt{(kv_1)^2 + (kv_2)^2}$

$= \sqrt{k^2 v_1{}^2 + k^2 v_2{}^2}$

$= \sqrt{k^2 (v_1{}^2 + v_2{}^2)}$

$= |k|\sqrt{v_1{}^2 + v_2{}^2}$

$= |k|\,\|\mathbf{v}\|$

$= k\|\mathbf{v}\|.$

Example (12): If **u**=<4, 2, −4>, find the scalar k such that $\|k\mathbf{v}\|= 6$.

Solution:

$\|k\mathbf{v}\| = k\|\mathbf{v}\|= 6$, **v**= <4, 2, −4>

$\|\mathbf{v}\|= \sqrt{(4)^2 + (2)^2 + (-4)^2}$;

$= \sqrt{(16 + 4 + 16}$

$= \sqrt{36}$

$= 6 \longrightarrow 6k = 6 \longrightarrow k= 1$

Check: $k\|\mathbf{v}\|= 1(6)= 6$.

Example (13): If **v** is any nonzero vector, then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector. (Show that).

Proof:

The magnitude of a unit vector is 1, thus we must prove that the magnitude of $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is $1(\left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\|= 1)$.

Since $\frac{1}{\|\mathbf{v}\|}$ is a scalar, then

$\left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\|= \left|\frac{1}{\|\mathbf{v}\|}\right| \|\mathbf{v}\|= \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\|= 1.$

$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector.

Example (14): Find a unit vector that has the same direction as the vector **v**=<3, 4>.

Solution:

$\mathbf{U}= \frac{1}{\|\mathbf{v}\|}\mathbf{v}, \ \|\mathbf{v}\|= \sqrt{(3)^2 + (4)^2}= 5$

$\mathbf{U}= \frac{1}{5}$<3, 4>

U= $<\frac{3}{5}, \frac{4}{5}>$ the unit vector. (We can check that: $\|\mathbf{U}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$).

Note: If we want to find the unit vector that has the opposite direction for any vector like **v,** this means finding the unit vector for the vector (–**v**).

Example (15): Find the unit vector that has the opposite direction of the vector **v=<-4**, 2, 4>.

(Homework).

Example (16): Find the unit vector of the vector **v**= <5, -2, 1> when the angle between these two vectors is zero. **(Homework)**.

The distance d between two points in 2-space or 3-space:

If $P_1= (x_1, y_1, z_1)$ and $P_2= (x_2, y_2, z_2)$ are two points in 3-space as shown in **(figure 21)**, then the distance d between them is the norm (magnitude, length) of the vector $\overrightarrow{P_1 P_2}$, thus we must find the coordinates of this vector, then determine its magnitude or norm (the distance), as follows:

$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1);$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Similarly, for the vectors in 2-space, such that:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

when $P_1= (x_1, y_1)$ and $P_2= (x_2, y_2)$ are two points in 2-space.

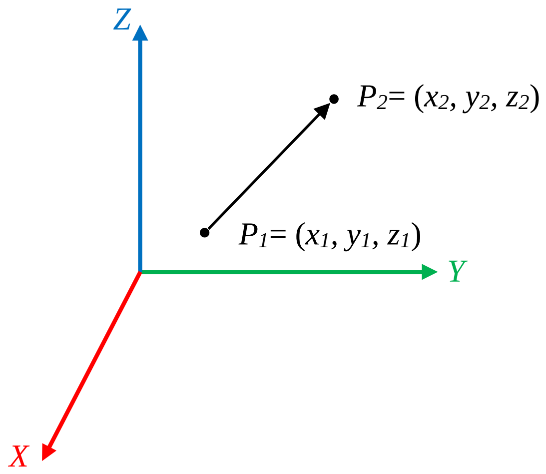

Figure 21

The distance between P_1 and P_2 is the norm of $\overrightarrow{P_1 P_2}$

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Example (17): find the norm (magnitude) of the vector **v=** <4, -1, 2>, and find the distance d between the points P_1= (3, -2, 3) and P_2= (5, -6, -5). **(Homework).**

Dot product of vectors:

Angle between vectors:

If **u** and **v** are two nonzero vectors in 2- or 3-space, and they have the same initial point, then the angle between **u** and **v** is denoted by θ and satisfies $0 \leq \theta \leq \pi$, as shown in **(figure 22)**.

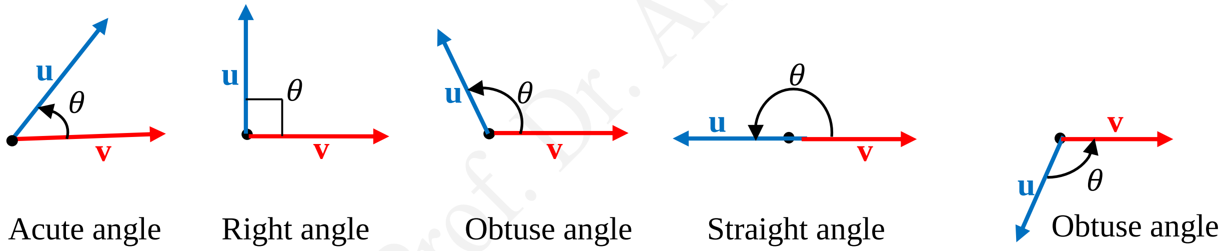

Figure 22

The angle θ between **u** and **v** satisfies $0 \leq \theta \leq \pi$

<u>**Definition (11)**</u>**: (dot product or Euclidean inner product of vectors):**

If **u** and **v** are two nonzero vectors in 2- or 3-space(or R^n-space) and θ is the angle between them, then the **dot product or Euclidean inner product u . v** is defined by:

$$\mathbf{u}\cdot\mathbf{v} = \begin{cases} \|\mathbf{u}\|\ \|\mathbf{v}\|\ \cos\theta & \text{if } \mathbf{u} \neq 0 \text{ and } \mathbf{v} \neq 0. \\ 0 & \text{if } \mathbf{u} = 0 \text{ or } \mathbf{v} = 0. \end{cases} \qquad \text{.......} \ (1)$$

The result of **u . v** *is a scalar.*

Example (18): If **u=** <6, -2, -3> and **v=** <1, 1, 1>, then find **u . v** when the angle θ between them is 85°.

Solution:

$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\|\ \cos\theta$

$$\|\mathbf{u}\| = \sqrt{6^2 + (-2)^2 + (-3)^2}$$
$$= \sqrt{49}$$
$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 1^2}$$
$$= \sqrt{3}$$
$$\rightarrow \quad \mathbf{u} \cdot \mathbf{v} = \sqrt{49}\sqrt{3} \cos 85^{\circ}$$
$$= \sqrt{147}\,\frac{1}{\sqrt{147}}$$
$$= 1.$$

Example (19): Find the dot product of the vectors **u**= <0, 0, 1> and **v**= <0, 2, 2> if the angle θ between them is 45°. **(Homework).**

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2- Elementary Linear Algebra Subsequent Edition, Arthur Wayne Roberts,1985.

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