Ministry of Higher Education
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lectures
Subject: <u>Vector analysis.</u>
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Stage: 2[st].
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Theorem (3): (Properties of the dot product):

If **u, v** and **w** are three vectors in 2-or 3-space, and let k is any scalar then:

a) **u . v = v . u (Commutative)**.
b) **u . (v + w) = u . v + u . w (Distributive property)**.
c) k(**u . v**) = (k**u**) **. v** = **u .** (k**v**) **(Associative)**.
d) **v .v** > 0 if **v** ≠ 0 and **v .v** = 0 if **v** = 0.

Proof (a):

Let **u**=< u_1, u_2, u_3> and **v**= < v_1, v_2, v_3> are tow vectors in 3-space then:

u . v= $(u_1v_1 + u_2v_2 + u_3v_3)$

= $(v_1u_1 + v_2u_2 + v_3u_3)$ (commutative property of multiplication)

= **v . u**

Proof (b):

Let **u**=< u_1, u_2, u_3>, **v**= < v_1, v_2, v_3> and **w**= < w_1, w_2, w_3> are three vectors in 3-space then:

u . (v + w)= < u_1, u_2, u_3> **.** < $v_1+ w_1$, $v_2+ w_2$, $v_3+ w_3$>

$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$

$= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3+ u_3w_3$

$= (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3)$

= **u . v** + **u . w**

Proof (c):

k (**u . v**)= $k(u_1v_1 + u_2v_2 + u_3v_3)$

$= (ku_1)v_1 + (ku_2)v_2 + (ku_3)v_3$

= (k**u**) . **v**, and similarly for k(**u . v**)= **u .** (k**v**). (<u>**Homework** part (d)).</u>

Parallel vectors:

We previously said that the two nonzero vectors are **parallel** if they have the same direction and the angle between these parallel vectors is zero degree, and if the angle between them is 180° then they are parallel and point in the **opposite direction**.

1- Two nonzero vectors in 2-or 3-space are **parallel** if they are scalar multiples of one another.

If **u** and **v** are two nonzero vectors and **u**= k**v** (k any scalar), then **u** and **v** are **parallel**.
For example**:** if **u**=<u_1, u_2, u_3> and **v**= <v_1, v_2, v_3>, then they are parallel if and only if:

$$\mathbf{u}= k\mathbf{v}= k<v_1, v_2, v_3>= <kv_1, kv_2, kv_3>.$$

2- Two nonzero vectors are parallel if and only if $|\mathbf{u}.\mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$ and they have the same direction if and only if $\mathbf{u}.\mathbf{v}= \|\mathbf{u}\|\|\mathbf{v}\|$.

That means they are parallel if $\cos\boldsymbol{\theta}= \mp\mathbf{1}$ $\left(\cos\theta = \dfrac{\mathbf{u}\,.\,\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$ and they have the same direction if $\cos\boldsymbol{\theta}= \mathbf{1}$ ($\cos\theta= 1 \rightarrow \theta= 0$, and $\cos\theta= -1 \rightarrow \theta= 180°$).

Example (29): Determine which of the following vectors are parallel to vector **v**= <−3, −2, 5>.

a) **u=** <6, 4, −10>;
b) **w=** <$\frac{-3}{2}$, -1, $\frac{-5}{2}$>;
c) **d**= <2, $\frac{4}{3}$, $\frac{-10}{3}$>.

Solution:

a) If **v**= k**u** then, they are parallel.

$<-3, -2, 5>= k<6, 4, -10>= <6k, 4k, -10k>$

$-3= 6k \rightarrow k= -\frac{1}{2}$, $-2= 4k \rightarrow k= -\frac{1}{2}$, $5= -10k \rightarrow k= -\frac{1}{2}$

$k\mathbf{u}= -\frac{1}{2}<6, 4, -10>= <-3, -2, 5>= \mathbf{v}$, so the two vectors are parallel.

We will obtain the same result if we take **u**= k**v**, *but* k= -2. (***check this***).

b) If **v**= k**w**, then they are parallel.

$<-3, -2, 5>= k<\frac{-3}{2}, -1, \frac{-5}{2}>= <\frac{-3}{2}\,k, -1k, \frac{-5}{2}\,k>$

$-3= \frac{-3}{2}\,k \;\rightarrow\; k= 2$, $-2= -1k \;\rightarrow\; k= 2$, $5= \frac{-5}{2}\,k \;\rightarrow\; k= -2.$

k is not equal in all three cases, so the two vectors are not parallel.

Now, if we use the second method:

$\|\mathbf{v}\| = \sqrt{(-3)^2+(-2)^2+(5)^2} = \sqrt{38}$;

$\|\mathbf{w}\| = \sqrt{(\frac{-3}{2})^2+(-1)^2+(\frac{-5}{2})^2} = \frac{\sqrt{38}}{2} \rightarrow \|\mathbf{v}\|\|\mathbf{w}\| = 19$;

$\mathbf{v} \cdot \mathbf{w} = <-3, -2, 5> \cdot <\frac{-3}{2}, -1, \frac{-5}{2}>$

$= \frac{9}{2} + 2 - \frac{25}{2}$

$= -6$

$|\mathbf{v}.\mathbf{w}| \neq \|\mathbf{v}\|\|\mathbf{w}\|$, so the two vectors are not parallel. (**Homework** part (c)).

Example (30): Find a vector **v** of length 2 units which is parallel and in same direction to **u**= <2, −2, 1>.

Solution: (**first method**).

We use the formula of unit vector to find the components of the vector **v.**

$\mathbf{U} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \rightarrow \mathbf{v} = \mathbf{U}\|\mathbf{v}\|$, $\|\mathbf{v}\| = 2$, we need to find **U**.

v is parallel to **u** and with the same direction as **u**, thus we find the unit vector from the vector u.

$\|\mathbf{u}\| = \sqrt{2^2+(-2)^2+1^2} = \sqrt{9} = 3$

$\mathbf{U} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{<2,-2,1>}{3} = <\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}>$

$\mathbf{v} = \mathbf{U}\|\mathbf{v}\|$

$= 2<\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}> = <\frac{4}{3}, \frac{-4}{3}, \frac{2}{3}>$ and we can check this by find $\|\mathbf{v}\|=2$.

Second method: Two vectors are **parallel** if they are scalar multiples of one another.

$\mathbf{v} = k\mathbf{u}$,

$= k<2, -2, 1> = <2k, -2k, k>$, $\|\mathbf{v}\| = 2$;

$\|\mathbf{v}\| = \sqrt{(2k)^2+(-2k)^2+k^2}$

$= \sqrt{9k^2}$

$= 3k \rightarrow \|\mathbf{v}\| = 3 \rightarrow 2 = 3k \rightarrow k = \frac{2}{3}$

So $\mathbf{v} = <\frac{4}{3}, \frac{-4}{3}, \frac{2}{3}>$.

Example (31): find the components of the vector **v**, that has the same direction as **u**= <3, 4>, and ||**v**||= 12. **(Homework)**.

Example (32): If ||**v**||= 10, and **u**= 8**i** +15**j,** find the components of the vector **v** if it has the same direction as **u**. **(homework)**.

How to find the magnitude and direction of a given vector?

Example (33): Find the magnitude and direction angle of the vector **v**= −2**i** + 5**j**.

Solution:

$\|\mathbf{v}\| = \sqrt{(-2)^2 + 5^2}$

$= \sqrt{29}$ the magnitude.

In **(figure 29)** we note that we must find the angle θ_1 (from X-axis to the vector), so first we must find the angle θ_2 (from $-X$-axis to the vector).

$\tan\theta_2 = \frac{5}{-2} \rightarrow \theta_2 = \tan^{-1}(\frac{5}{-2})$

$\approx 68.2°$

$\theta_1 = 180° - \theta_2$

$= 180° - 68.2°$

$= 111.8°$ the angle.

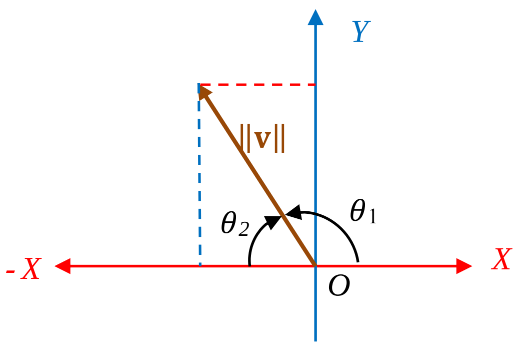

Figure 29

Example (34): Find the components of the vector **v** if ||**v**||= 10 and θ= 40°.

Solution:

Let **v**= <v_1, v_2> as shown in (**figure 30**), we must find v_1 and v_2.

$\cos 40° = {}^{v_1}/_{\|\mathbf{v}\|}$

$v_1 = \|\mathbf{v}\| \cos 40°$

$= 10(0.766)$

$= 7.66$

$\sin 40° = {}^{v_2}/_{\|\mathbf{v}\|}$

$v_2 = \|\mathbf{v}\| \sin 40°$

$= 10(0.642)$

$= 6.427$

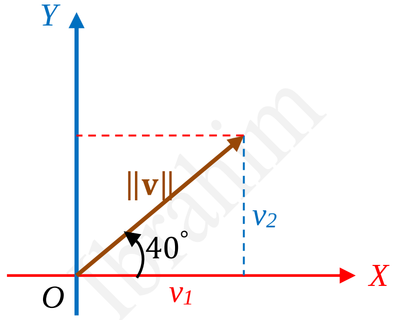

Figure 30

Thus **v**= <7.66, 6.427> or **v**= 7.66**i** + 6.427**j** (as linear combination form).

Example (35): Find a vector **v** in linear combination form if $\|\mathbf{v}\|$= 25 and θ= 146°. (**Homework**). (with drawing the figure).

Example (36): determine whether the two nonzero vectors **v**= -3**i** – 4**j** and **u**= 6**i** + 8**j**, are parallel and have the same direction. (**Homework**).

Example (37): Find the magnitude and direction of the vector **v**= 3(cos 30**i** + sin 30**j**). (**Homework**).

Example (38): Determine whether each pair of the following vectors are orthogonal, parallel or neither.

u= <1, 0, 0, 1>, **v**= <0, 1, 1, 0> and **w**= <3, 0, 0, 3>.

Solution:

u . v= 0 → **u** and **v** are orthogonal.

v . w= 0 → **v** and **w** are orthogonal.

u . w= 6 → **u** and **w** are not orthogonal.

Now we check whether the two vectors **u** and **w** are parallel.

u . w= 6, $\|\mathbf{u}\|=\sqrt{2}$, $\|\boldsymbol{w}\|=\sqrt{18}$, $\|\mathbf{u}\|\|\mathbf{w}\|=6$;

Since **u . w**= $\|\mathbf{u}\|\|\mathbf{w}\|$ → **u** and **w** are parallel and in the same direction.

Note: If **u . w**= −6 then $|\mathbf{u}\cdot\mathbf{w}|=\|\mathbf{u}\|\|\mathbf{w}\|$ and the vectors **u** and **w** are parallel but not in the same direction.

Example (39): Determine whether each pair of the following vectors are orthogonal, parallel (in the same direction) or neither. (**a, b, c**, and **e Homework**).

a) **v**= <−5, 3, 7>, **u**= <6, −8, 2>;
b) **v**= <4, 6>, **u**= <−3, 2>;
c) **v**= −**i** + 2**j** +5**k**, **u**= 3**i** + 4**j** −**k**;
d) **v**= 2**i** + 6**j** −4**k**, **u**= −3**i** −9**j** + 6**k**;
e) **v**= −3**i** −4**j**, **u**= 6**i** + 8**j**.

Solution(d):

v . u= (2)(−3) + (6)(−9) + (−4)(6)

= −84;

$\|\mathbf{v}\|=\sqrt{56}$, $\|\mathbf{u}\|=\sqrt{126}$ → $\|\mathbf{v}\|\|\mathbf{u}\|=84$

Since $|\mathbf{v}.\mathbf{u}|=\|\mathbf{v}\|\|\mathbf{u}\|$ → the vectors are parallel but are not in the same direction.

References

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2- Elementary Linear Algebra Subsequent Edition, Arthur Wayne Roberts,1985.

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4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.