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Application of the scalar triple product:

Theorem (6):

a) The absolute value of the determined $\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$ is equal to the **area of the parallelogram** in 2-space determined by the vectors **u**= <u_1, u_2> and **v**= <v_1, v_2>, as shown in figure (41).

b) The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$ is equal to the **volume of the parallelepiped** determined by the vectors **u**=<u_1, u_2, u_3>, **v**= <v_1, v_2, v_3> and **w**= <w_1, w_2, w_3>, as shown in figure (42).

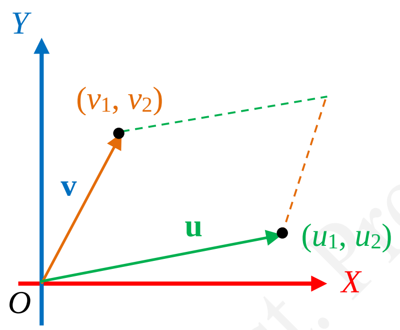

Figure 41

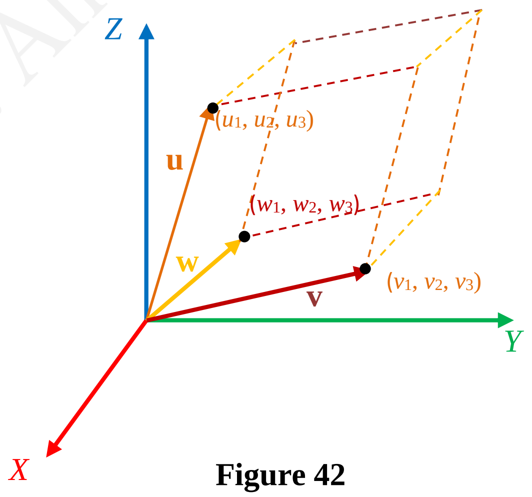

Figure 42

Proof (a): To prove this part we can use theorem (5) { if **u** and **v** are vectors in 3-space, then $\|\mathbf{u} \times \mathbf{v}\|$ is equal to the area of the parallelogram determined by **u** and **v**}, but this theorem applies to vectors in 3-space and **u**= <u_1, u_2>, **v**= <v_1, v_2> are vectors in 2-space as shown in figure (41) above, thus to avoid this problem we will write these vectors **u** and **v** as vectors in 3-space **u**= < u_1, u_2, 0>, **v**= < v_1, v_2, 0> as shown in figure (43) below.

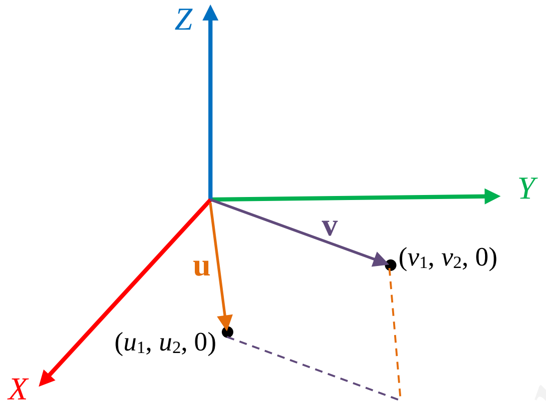

Figure 43

Now we must find the area of the parallelogram determined by these vectors to show it is equal to the absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} = \det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\mathbf{k}.$$

By theorem (5), the area A_P *is* $\|\mathbf{u} \times \mathbf{v}\|$.

Thus, $A_P = \|\mathbf{u} \times \mathbf{v}\| = \left\|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\mathbf{k}\right\| = \left|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right| \|\mathbf{k}\|$

Since $\|\mathbf{k}\| = 1$, then $\|\mathbf{u} \times \mathbf{v}\| = \left|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|$.

Proof (b):

The volume of the parallelepiped is V= (area of the base) (height).

The parallelepiped determined by the vectors **u**, **v**, and **w**, and the base of this parallelepiped is parallelogram determined by **v** and **w** as shown in figure (42) above, and the area of this parallelogram is $\|\mathbf{v} \times \mathbf{w}\|$ (theorem 5).

The height (h) of the parallelepiped is the length of the orthogonal projection of **u** along **v**×**w**, as shown in figure (44) below, now we can write the height h as:

$$\|\mathrm{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \frac{|\mathbf{u} \cdot (\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}$$

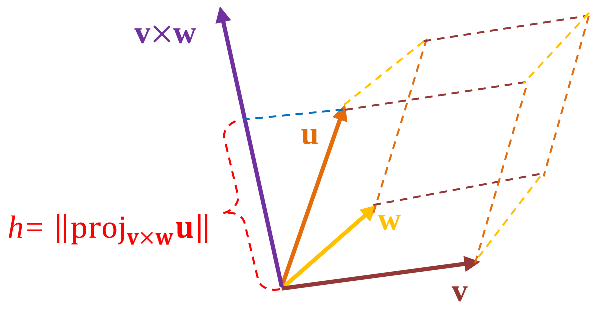

Figure 44

Thus, the volume of the $V = \|\mathbf{v}\times\mathbf{w}\|\,\|\mathrm{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|$

$$= \|\mathbf{v}\times\mathbf{w}\|\,\frac{|\mathbf{u}\,.(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}$$

$$= |\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w})| = V.$$

Since, $\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w})$ is the scalar triple product, and

$$\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

Then, $V = |\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w})| = \left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right|.$

Example (54): Find the volume of the parallelepiped, that is determined by the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Solution:

$$V = |\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w})|$$

$$\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w} = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= -10$$

$V = |\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})| = |-10| = 10$ the volume.

Note: Also we can solve as the following:

1) Find $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 5\mathbf{i} - 5\mathbf{k} = \langle 5, 0, -5 \rangle$.
2) $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \langle 1, -2, 3 \rangle \,.\, \langle 5, 0, -5 \rangle$
   $= (1)(5) + (-2)(0) + (3)(-5)$
   $= -10.$
3) $V = |\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})| = |-10| = 10$ the volume.

Theorem (7): If the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, have the same initial point, then they lie in the same plane if and only if,

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

Example (55): Determine whether the vectors **u**, **v**, and **w** lie in the same plane, when positioned so that their initial points are coincide.

a) $\mathbf{u} = \langle -1, -2, 1 \rangle$, $\mathbf{v} = \langle 3, 0, -2 \rangle$ and $\mathbf{w} = \langle 5, -4, 0 \rangle$.
b) $\mathbf{u} = \langle 5, -2, 1 \rangle$, $\mathbf{v} = \langle 4, -1, 1 \rangle$ and $\mathbf{w} = \langle 1, -1, 0 \rangle$.
c) $\mathbf{u} = \langle 4, -8, 1 \rangle$, $\mathbf{v} = \langle 2, 1, -2 \rangle$ and $\mathbf{w} = \langle 3, -4, 12 \rangle$.

Solution (a):

$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} -1 & -2 & 1 \\ 3 & 0 & -2 \\ 5 & -4 & 0 \end{vmatrix} = 16$, then the vectors do not in the same plane.

Theorem (8) (Cauchy-Schwarz inequality in R^n(n-space)):

For any vectors **u** and $\mathbf{v} \in R^n$, $|\mathbf{u} \,.\, \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$.

If $\mathbf{u} = 0$ (or $\mathbf{v} = 0$), then $\|\mathbf{u}\| = 0$ and $\mathbf{u} \,.\, \mathbf{v} = 0$, therefore the inequality is satisfying.

Some information that helps us understand the proof of this theorem.

$R^n = \{(x_1, x_2, \ldots, x_n): x_i \in \mathbb{R}\}$or $R^n = \left\{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}: x_{i \in \mathbb{R}}\right\}$.

Let $\mathbf{u}= <u_1, u_2, \ldots, u_n>$ and $\mathbf{v}= < v_1, v_2, \ldots, v_n>$, then

$$\mathbf{u} \cdot \mathbf{v}= u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

$$= \sum_{\mathrm{i}}^{\mathrm{n}} u_{\mathrm{i}} v_{\mathrm{i}}$$

$\mathbf{u} \cdot \mathbf{u}= u_1^2 + u_2^2 + \ldots + u_n^2= \|\mathbf{u}\|^2 \rightarrow \|\mathbf{u}\|= \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

If $\mathbf{u}$, $\mathbf{v}$ are two nonzero vectors $\in R^2$ or R^3 and θ the angle between them, then

$\mathbf{u} \cdot \mathbf{v}= \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$ and $\cos\theta= \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \rightarrow \theta= \cos^{-1}\dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

since $-1 \leq \cos\theta \leq 1$, then $-1 \leq \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1$ and $\leq 1 \leftrightarrow \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1$

$$\leftrightarrow |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Example (56): If $\mathbf{u}= <4, 2, 1, 3>$ and $\mathbf{v}= <2, 3, 2, -1>$, show that $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$.

Solution:

$\mathbf{u} \cdot \mathbf{v}= 13$ and $|\mathbf{u} \cdot \mathbf{v}|= 13$.

$\|\mathbf{u}\|= \sqrt{30}$, $\|\mathbf{v}\|= \sqrt{18} \rightarrow 13 < \sqrt{30}\sqrt{18}$.

Theorem (9) (The triangle inequality theorem).

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors in R^n, then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

To prove this theorem, we must first remember some important facts related to this theorem.
The sum of any two vectors (**vector addition**) represents the vector that began from the initial point of one of these vectors to the terminal point of the another vector, therefore we will obtain a triangle in which the following characteristic is achieved, which is that the sum of the lengths of any two sides in this triangle is greater than the third side ($\|\mathbf{u} + \mathbf{v}\| < \|\mathbf{u}\| + \|\mathbf{v}\|$) figure (45).

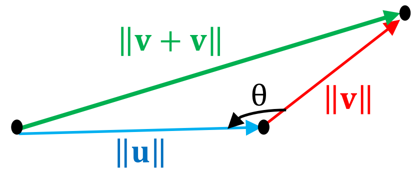

Figure 45

Now we discuss when this inequality will be, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

We know that the angle θ ($0 \leq \theta \leq \pi$) between the two vectors as shown in figure (45), and therefore, the length of the side (vector) corresponding to the angle is smaller when the angle is smaller and increases when the angle becomes larger so that it becomes equal to the sum of the lengths of the others two sides (vectors) when the angle becomes 180°, then the vectors are parallel ($\mathbf{u}= k\mathbf{v}$, k is any scalar $\in \mathbb{R}^+$) and $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ as shown in figure (46), therefore in general **The triangle inequality** ($\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$) is correct according to the angle between the two vectors.

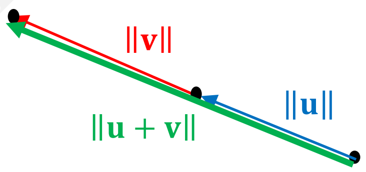

Figure 46
$\mathbf{u}= k\mathbf{v}$

Proof:

$$\|\mathbf{u} + \mathbf{v}\|^2= (\mathbf{u} + \mathbf{v}) \,.\, (\mathbf{u} + \mathbf{v}) \qquad \{\|\mathbf{v}\|^2= v_1^2 + v_2^2= \mathbf{v} \,.\, \mathbf{v}\}$$

$$= \mathbf{u} \,.\, (\mathbf{u} + \mathbf{v}) + \mathbf{v} \,.\, (\mathbf{u} + \mathbf{v})$$

$$= \mathbf{u} \,.\mathbf{u} + \mathbf{u} \,.\mathbf{v} + \mathbf{u} \,.\, \mathbf{v} + \mathbf{v} \,.\, \mathbf{v} \qquad (\mathbf{u} \,.\, \mathbf{v}= \mathbf{v} \,.\, \mathbf{u}, \text{ Commutative})$$

$$= \mathbf{u} \,.\, \mathbf{u} + 2(\mathbf{u} \,.\, \mathbf{v}) + \mathbf{v} \,.\, \mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u} \,.\, \mathbf{v}) + \|\mathbf{v}\|^2$$

Now by theorem (8) (Cauchy-Schwarz inequality in R^n(n-space))

$$|\mathbf{u} \,.\, \mathbf{v}| \leq ||\mathbf{u}||\,||\mathbf{v}||;$$

Thus, $\quad ||\mathbf{u}||^2 + \underline{2(\mathbf{u} \,.\, \mathbf{v})} + ||\mathbf{v}||^2 \leq ||\mathbf{u}||^2 + \underline{2||\mathbf{u}||\,||\mathbf{v}||} + ||\mathbf{v}||^2.$

The question now is how we used the Cauchy-Schwarz inequality in this case, whereas this inequality includes the absolute value of the dot product for two vectors $|\mathbf{u} \,.\, \mathbf{v}|$ and not just the dot product ($\mathbf{u} \,.\, \mathbf{v}$).

The answer is, that we discuss the situation when the two vectors are parallel.

So, $\mathbf{u}= k\mathbf{v}$ and $\mathbf{u} \,.\, \mathbf{v}= k\mathbf{v} \,.\, \mathbf{v}= k||\mathbf{v}||^2$, k is positive and $||\mathbf{v}||^2 > 0$.

Thus, $\quad \mathbf{u} \,.\, \mathbf{v}= |\mathbf{u} \,.\, \mathbf{v}|.$

And, $\quad ||\mathbf{u} + \mathbf{v}||^2 \leq (||\mathbf{u}|| + ||\mathbf{v}||)^2$

Now by square root for two sides we obtain:

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

The general meaning of this theory is that the length of any side in a triangle does not exceed the sum of the lengths of the other two sides, as shown by the following examples.

Example (57): Let $\mathbf{x}= <1, 0, 0, 1>$ and $\mathbf{y}= <0, 1, 1, 0>$, show that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

Solution:

$\mathbf{x} + \mathbf{y}= <1, 1, 1, 1>$

$||\mathbf{x} + \mathbf{y}||= \sqrt{1^2 + 1^2 + 1^2 + 1^2}= \sqrt{4}= 2$

$||\mathbf{x}||= \sqrt{2}$, $||\mathbf{y}||= \sqrt{2}$

And $2 < \sqrt{2} + \sqrt{2}$, therefore $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

In general, this means that if we have the following triangle (figure 47), then according to this theorem, the sides should be as follows:

$\overline{AB} + \overline{BC} > \overline{AC} \to 5 + 3 > 4;$

$\overline{BC} + \overline{AC} > \overline{AB} \to 3 + 4 > 5;$

$\overline{AC} + \overline{AB} > \overline{BC} \to 4 + 5 > 3.$

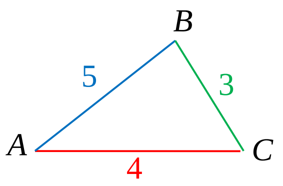

Figure 47

Example (58): Determine the possible length of **x** in the following triangle (figure 48).

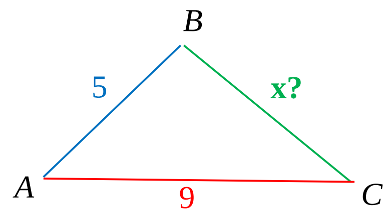

Figure 48

Solution:

$\overline{AB} + \overline{BC} > \overline{AC} \rightarrow 5 + \mathbf{x} > 9 \rightarrow \mathbf{x} > 4$ … (1)

$\overline{BC} + \overline{AC} > \overline{AB} \rightarrow \mathbf{x} + 9 > 5 \rightarrow \mathbf{x} > -4$, and this is not possible, because $\|\boldsymbol{x}\| \geq 0$.

$\overline{AC} + \overline{AB} > \overline{BC} \rightarrow 9 + 5 > \mathbf{x} \rightarrow 14 > \mathbf{x}$ … (2)

Thus, its true if and only if $4 < \mathbf{x} < 14$.

Example (59): Find the area of the parallelogram determined by the vectors **u**= <2, 4> and **v**= <6< 1>. (**Homework**).

Example (60): Find the area of the tringle determined by the points P_1= (1, 2, 2), P_2= (0, 5, 3) and P_3= (0, 1, 5), and also find the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. (**Homework**).

References

1- Introductory linear algebra with applications, Bernard Kolman, first edition, 1976.

2- Elementary Linear Algebra Subsequent Edition, Arthur Wayne Roberts,1985.

3- Elementary Linear Algebra, Ninth Edition, Howard Anton, Chris Rorres, 2005.

4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.